Finite Element Heterogeneous Multiscale Method for the Wave Equation: Long-Time Effects

Assyr Abdulle[1], **Marcus J. Grote**[2], **Christian M. Stohrer**[2,*]
[1] ANMC, Section of Mathematics, EPFL, Lausanne, Switzerland
[2] Department of Mathematics and Computer Science, University of Basel, Basel, Switzerland
*Email: christian.stohrer@unibas.ch

Abstract

For limited time the propagation of waves in a highly oscillatory medium is well-described by the non-dispersive homogenized wave equation. With increasing time, however, the true solution deviates from the classical homogenization limit, as a large secondary wave train develops unexpectedly. Here, we propose a new finite element heterogeneous multiscale method (FE-HMM), which captures not only the short-time macroscale behavior of the wave field but also those secondary long-time dispersive effects.

1 Long-Time Wave Propagation

Let $\Omega \subset \mathbb{R}^n$ be a domain and $T > 0$. We consider the wave equation

$$\begin{cases} \partial_{tt} u^\varepsilon - \nabla \cdot (a^\varepsilon \nabla u^\varepsilon) = F & \text{in } \Omega \times (0,T), \\ u^\varepsilon(x,0) = f(x) & \text{in } \Omega, \\ \partial_t u^\varepsilon(x,0) = g(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $a^\varepsilon(x) \in (L^\infty(\Omega))^{d\times d}$ is symmetric, uniformly elliptic, and bounded. Here $\varepsilon > 0$ represents a small scale in the problem, which characterizes the multiscale nature of the tensor $a^\varepsilon(x)$. We set either homogeneous Dirichlet or periodic boundary conditions to uniquely determine the solution for every $\varepsilon > 0$.

1.1 Classical homogenization

According to classical homogenization theory, u^ε converges to the solution u^0 of the "homogenized" wave equation as $\varepsilon \to 0$,

$$\partial_{tt} u^0 - \nabla \cdot (a^0 \nabla u^0) = F,$$

where the homogenized tensor (or squared velocity field) a^0 can only rarely be computed explicitly. Thus, u^0 approximates u^ε but only for short times. For longer times $T \sim \varepsilon^{-2}$, the homogenized solution becomes increasingly inadequate, since it neglects microscopic dispersive effects that accumulate over time, as shown in Figure 1. Here we consider (1) in $\Omega = (-1,1)$ with periodic boundary conditions, let $u(x,0)$ be a Gaussian pulse with zero initial velocity

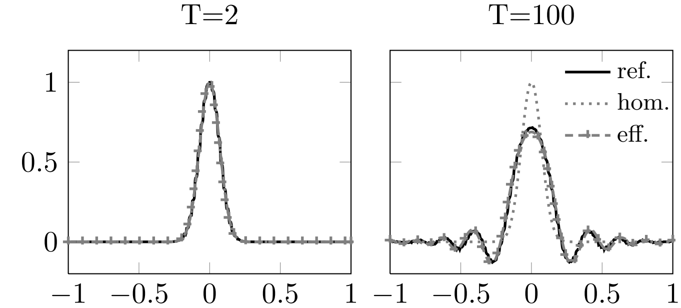

Figure 1: Reference (ref.), homogenized (hom.) and effective (eff.) solution: short-time (left) and long-time (right).

and set

$$a^\varepsilon = \sqrt{2} + \sin\Big(2\pi \frac{x}{\varepsilon}\Big) \text{ with } \varepsilon = \frac{1}{50}. \tag{2}$$

The reference solution of (1)–(2) corresponds to a direct numerical simulation (DNS), where the microscale is fully resolved. After one revolution ($T = 2$), the homogenized and the DNS solution coincide. After fifty revolutions ($T = 100$), however, the DNS displays dispersive effects, which the homogenized solution fails to capture.

1.2 Effective dispersive equation

Various formal asymptotic arguments were derived to elucidate that peculiar inherently dispersive long-time behavior of waves propagating through a strongly heterogeneous periodic medium [1]. An effective equation that captures those dispersive effects was recently derived in [2] for the one-dimensional case when a^ε is ε-periodic:

$$\partial_{tt}(u^{\text{eff}} - \varepsilon^2 b \partial_{xx} u^{\text{eff}}) - a^0 \partial_{xx} u^{\text{eff}} = F. \tag{3}$$

Again, a^0 denotes the homogenized effective coefficient from classical homogenization theory and $b > 0$. As shown in Figure 1, u^ε and u^{eff} essentially coincide both at early and later times.

2 FE Heterogeneous Multiscale Method

In [3], the FE-HMM for elliptic [4] was extended to the time dependent wave equation. It was shown to converge to u^0 at finite times, yet it failed to capture long-time dispersive effects in the true solution. To incorporate those dispersive effects, we not only need an effective bilinear form but also an effective inner product, akin to the weak formulation of (3). Both require the numerical solutions of micro problems on sampling domains I_δ of size δ proportional to ε. An alternative HMM scheme, based on the finite difference approximation of an effective flux, was proposed in [5].

We now give a description of the algorithm: First, we generate a macro triangulation $\mathcal{T}_H$ and choose an appropriate macro FE space $S(\Omega, \mathcal{T}_H)$. By macro we mean that $H \gg \varepsilon$ is allowed. Within each macro element $K \in \mathcal{T}_H$ we choose a quadrature formula $\{x_{K,j}, \omega_{K,j}\}$. The HMM solution u_H is given by the following variational problem:

$$\begin{cases} \text{Find } u_H : [0,T] \to S(\Omega, \mathcal{T}_H) \text{ such that} \\ (\partial_{tt} u_H, v_H)_H + B_H(u_H, v_H) = (F, v_H) \\ \text{for all } v_H \in S(\Omega, \mathcal{T}_H) \text{ and,} \\ \quad u_H(0) = f_H,\ \partial_t u_H(0) = g_H \text{ in } \Omega, \end{cases} \tag{4}$$

where the initial data f_H and g_h are suitable approximations of f and g in $S(\Omega, \mathcal{T}_H)$, $(\cdot,\cdot)_H$ and B_H are defined below. Since $(\cdot,\cdot)_H$ is an inner product and the bilinear form B_H is elliptic and bounded, the FE-HMM is well defined.

The FE-HMM inner product is defined by

$$(v_H, w_H)_H := \sum_{K,j} \frac{\omega_{K,j}}{|I_\delta|} \cdot \int_{I_\delta} (v_H(x_{K,j}) + v_h(y))(w_H(x_{K,j}) + w_h(y))\, dy,$$

and the FE-HMM bilinear form by

$$B_H(v_H, w_H) := \sum_{K,j} \frac{\omega_{K,j}}{|I_\delta|} \cdot \int_{I_\delta} a^\varepsilon(y)(\nabla v_H(x_{K,j}) + \nabla v_h) \cdot (\nabla w_H(x_{K,j}) + \nabla w_h)\, dy,$$

where v_h (resp. w_h) is the solution of the micro problem

$$\begin{cases} \text{Find } v_h \in S(I_\delta, \mathcal{T}_h) \text{ such that} \\ \displaystyle\int_{I_\delta} a^\varepsilon(y)(\nabla v_H(x_{K,j}) + \nabla v_h(y)) \cdot \nabla z_h(y)\, dy = 0, \\ \text{for all } z_h \in S(I_\delta, \mathcal{T}_h). \end{cases} \tag{5}$$

Here $S(I_\delta, \mathcal{T}_h)$ is a micro FE space on the sampling domain I_δ with a micro triangulation $\mathcal{T}_h$.

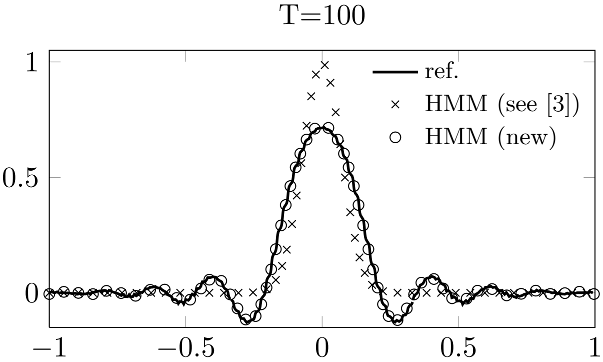

Figure 2: Reference solution (ref.), FE-HMM from [3] and new FE-HMM.

3 Numerical Experiments

We again apply our FE-HMM, defined in (4), to (1)–(2) as in Figure 1. We use cubic FE at the macro- and the micro-scale, with mesh sizes $H = 1/75$ and $h = \varepsilon/20 = 1/1000$. Note that linear or quadratic finite elements could also be used. For time-stepping we use a standard Leap-Frog scheme, with $\Delta t = H/10$. As shown in Figure 2, the new FE-HMM succeeds in capturing, the long-time effects in the true solution. In contrast, the solution of the FE-HMM of [3] is unable to capture those dispersive effects, since this solution was proven to converge to the homogenized solution, u^0, as $\varepsilon \to 0$ on finite time intervals.

References

[1] F. Santosa and W. W. Symes, *A Dispersive Effective Medium for Wave Propagation in Periodic Composites*, SIAM J. Appl. Math., **51**, pp. 984–1005.

[2] A. Lamacz, *Dispersive Effective Models for Waves in Heterogeneous Media*, Math. Models Methods Appl. Sci., **21** (2011), pp. 1871–1899.

[3] A. Abdulle and M. J. Grote, *Finite Element Heterogeneous Multiscale Method for the Wave Equation*, Multiscale Model. Simul., **9** (2011), pp. 766–7921.

[4] A. Abdulle, *The Finite Element Heterogeneous Multiscale Method: a computational strategy for multiscale PDEs*, GAKUTO Internat. Seri. Math. Sci. Appl., **31** 2009, pp. 133–182.

[5] B. Engquist, H. Holst and O. Runborg , *Multiscale methods for wave propagation in heterogeneous media*, Comm. Math. Sci., **9** 2011, pp. 33–56.