2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Parametric Sensitivity of Path-Constrained Optimal Control: Towards Selective Input Adaptation

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***Abstract*— In the context of dynamic optimization, plant variations necessitate adaptation of the input profiles in order to guarantee both feasible and optimal operation. For those problems having path constraints, two sets of directions can be distinguished in the input space at each time instant: the so-called sensitivity-seeking directions, along which a small input variation does not affect the active path constraints; the complementary constraint-seeking directions, along which a variation affects the path constraints. Hence, three selective input adaptations are possible, namely, adaptation along each set of input directions and adaptation of the switching times between arcs. This paper considers parametric variations around a nominal optimal solution and quantifies the influence of these variations on each type of input adaptation.**

***Index Terms*— Parametric optimal control, change in optimal inputs, second-order sufficient conditions, constraint-seeking and sensitivity-seeking directions, change in switching times.**

I. INTRODUCTION

Optimal operation of dynamic processes subject to operational constraints falls into the scope of dynamic optimization. Dynamic optimization is useful in the process industry for reducing production costs, improving product quality, and meeting strict safety requirements and environmental regulations.

In nominal dynamic optimization, the uncertainty in process operation is discarded. That is, the optimal input profiles are calculated off-line using numerical optimization and are applied to the process in an open-loop manner. Such a process operation can be highly sensitive to process uncertainty, which includes plant-model mismatch, process disturbances and variations in initial conditions. This situation is complicated by the fact that a sufficiently accurate process model can rarely be obtained in practice.

This paper considers plant-model mismatch in the form of parametric variations. If some parameters deviate from their nominal values, a change in optimal inputs is required to maintain optimality and meet operational constraints. In practice, it is rarely possible to adapt all parts of optimal input profiles, nor is this necessary from a performance viewpoint. Highest priority should be given to the selective adaptation that results in acceptable performance loss compared to optimal operation of the perturbed process.

These considerations provide a strong motivation for the development of selective input-adaptation strategies. For problems comprising path constraints, possibly of the mixed input-state type, this paper proposes the novel concept of distinguishing, at each time instant, between two sets of directions in input space: the so-called *sensitivity-seeking* directions, along which a small input variation causes *no* change in the value of the active path constraints, and the directions orthogonal to them, called *constraint-seeking* directions, along which a variation does affect the path constraints. This way, three complementary input adjustments are obtained, namely, adaptation along each set of input directions and adaptation of the switching times between successive arcs (in the presence of multiple arcs). An important contribution of this paper concerns the derivation of quantitative expressions for the three types of input variations caused by parametric variations.

A possible application of these results is in the field of a recently developed methodology called NCO tracking [1], which converts a constrained optimization problem into a feedback control problem and uses process measurements to enforce optimality. Because not all parts of the input profiles can or need to be adapted, the ability to assess selective-adaptation strategies is indeed key in developing practical NCO-tracking controllers.

The outline of the paper is as follows. The mathematical formulation of the parametric optimal control problem is given in Section II, which also presents the necessary conditions of optimality. In Section III, the sensitivity- and constraint-seeking directions are defined. Quantitative expressions of the optimal input variations in the constraint- and sensitivity-seeking directions and in the arc switching times, caused by small parametric variations around a nominal solution, are derived in Sections IV and V, respectively. Section VI presents an illustrative example. Finally, Section VII concludes the paper and proposes future research directions.

II. PARAMETRIC OPTIMAL CONTROL PROBLEM

The following parametric optimal control problem in the parameters $\boldsymbol{\theta}$, subject to the mixed control-state inequality path constraints $\boldsymbol{\Omega}$, with given initial time t_0 and terminal time $t_{\rm f}$, is considered (OC($\boldsymbol{\theta}$)):

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\theta}); \quad \mathbf{x}(t_0) = \mathbf{h}(\boldsymbol{\theta}), \tag{1}$$

$$\Omega_i(t, \mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\theta}) \leq 0, \quad i \in \mathcal{I}_{\boldsymbol{\Omega}}, \tag{2}$$

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$$\min_{\mathbf{u}} \ \psi(t_\mathrm{f}, \mathbf{x}(t_\mathrm{f}), \boldsymbol{\theta}) + \int_{t_0}^{t_\mathrm{f}} \phi(t, \mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\theta}) dt, \quad (3)$$

where $\mathbf{u}(t) \in \mathbb{R}^{n_\mathbf{u}}$, $\mathbf{x}(t) \in \mathbb{R}^{n_\mathbf{x}}$, and $\mathcal{I}_\mathbf{\Omega} := \{1, \ldots, n_\mathbf{\Omega}\}$.

Let the nominal values of the system parameters be $\boldsymbol{\theta}_0$, and let $(\mathbf{u}^*, \mathbf{x}^*)$ be an optimal pair for the problem OC($\boldsymbol{\theta}_0$). The following first-order necessary conditions of optimality must hold almost everywhere (a.e.) in $[t_0, t_\mathrm{f}]$ [2]:

$$\mathbf{0} = \mathcal{H}_\mathbf{u}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\mu}^*(t), \boldsymbol{\theta}_0), \quad (4)$$

where $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ are vector functions satisfying:

$$\begin{aligned} \dot{\boldsymbol{\lambda}}^*(t) &= -\mathcal{H}_\mathbf{x}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\mu}^*(t), \boldsymbol{\theta}_0), \quad (5)\\ \boldsymbol{\lambda}^*(t_\mathrm{f}) &= \psi_\mathbf{x}(t_\mathrm{f}, \mathbf{x}^*(t_\mathrm{f}), \boldsymbol{\theta}_0),\\ 0 &= \mu_i^*(t) \Omega_i(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\theta}_0), \quad \forall i \in \mathcal{I}_\mathbf{\Omega}, \quad (6) \end{aligned}$$

and the Hamiltonian function, $\mathcal{H}$, is defined as

$$\begin{aligned} \mathcal{H}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\theta}) := \phi(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\theta}) + \boldsymbol{\lambda}^T \mathbf{f}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\theta})\\ + \boldsymbol{\mu}^T \mathbf{\Omega}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\theta}). \end{aligned}$$

The key assumptions throughout the paper are that (i) the nominal optimal inputs $\mathbf{u}^*$ consist of finitely many arcs, and (ii) the second-order sufficiency conditions (SOSC) for differentiability of the optimal solution, as described in [3] and [4], hold. In particular, the Hamiltonian function is assumed to be *regular*, which implies that the optimal inputs $\mathbf{u}^*$ are continuous in $[t_0, t_\mathrm{f}]$.

During process operation, the value of the system parameters can deviate from their nominal values $\boldsymbol{\theta}_0$. To compensate for the effect of such variations, it becomes necessary to adapt the input profiles in such a way that they satisfy the optimality conditions for the perturbed problem.

III. Constraint- and Sensitivity-seeking Directions

In this section, we characterize the constraint- and sensitivity-seeking directions in input space for small perturbations in the neighborhood of the nominal optimal inputs.

Let the structure of the nominal optimal inputs be such that the constraint Ω_i is active on N_i disjoint intervals $[a_{ik_i}, b_{ik_i}] \subset [t_0, t_\mathrm{f}]$, $\forall k_i \in \{1, \ldots, N_i\}$. Hence, for $i \in \mathcal{I}_\mathbf{\Omega}$ and for all $t \in T_i := \{[a_{i1}, b_{i1}], \ldots, [a_{iN_i}, b_{iN_i}]\}$, we have:

$$\Omega_i(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\theta}_0) = 0.$$

The time instants a_{ik_i} and b_{ik_i} are labeled the *switching times* associated with the constraint Ω_i.

Note that there might be overlap between some (or all) of the intervals $\{[a_{i1}, b_{i1}], \ldots, [a_{iN_i}, b_{iN_i}]\}$ for various constraints $i \in \mathcal{I}_\mathbf{\Omega}$. Hence, different numbers of constraints may be active at different time instants. The vector of active constraints at time t will be denoted by $\mathbf{\Omega}^a(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\theta}_0)$.

When the number of active constraints at time t, $n_{\mathbf{\Omega}^a}(t)$, is larger than zero but less than the number of inputs $n_\mathbf{u}$, two sets of directions can be distinguished in the input space depending on whether or not the active constraints are modified by taking an infinitesimal step along these directions.

Definition 1 (Sensitivity- and Constraint-seeking Directions): A direction in the input space $\mathbb{R}^{n_\mathbf{u}}$ along which an infinitesimal variation from the nominal inputs $\mathbf{u}^*(t)$ at a given time t does not modify the active constraints $\mathbf{\Omega}^a(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\theta}_0)$ is called a *sensitivity-seeking direction at* t. All the remaining directions in $\mathbb{R}^{n_\mathbf{u}}$ orthogonal to the set of sensitivity-seeking directions are called *constraint-seeking directions at* t.

From the regularity condition in SOSC, we see that the Jacobian matrix $\mathbf{\Omega}_\mathbf{u}^a$ must be full rank at $\mathbf{u}^*(t)$ a.e. in $[t_0, t_\mathrm{f}]$. As proposed in [5], the *singular value decomposition* (SVD) of the Jacobian matrix $\mathbf{\Omega}_\mathbf{u}^a$ can be used to compute the constraint- and sensitivity-seeking directions:

$$\mathbf{\Omega}_\mathbf{u}^a(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\theta}_0) = \mathbf{U}(t)\mathbf{\Sigma}(t)\mathbf{V}(t)^T,$$

where $\mathbf{\Sigma}(t)$ is a $(n_{\mathbf{\Omega}^a} \times n_\mathbf{u})$ matrix function of the form

$$\mathbf{\Sigma}(t) = \begin{bmatrix} \mathbf{\Sigma}_c(t) & \mathbf{0}_{n_{\mathbf{\Omega}^a} \times (n_\mathbf{u} - n_{\mathbf{\Omega}^a})} \end{bmatrix}$$

with

$$\mathbf{\Sigma}_c(t) := \begin{bmatrix} \sigma_1(t) & & \\ & \ddots & \\ & & \sigma_{n_{\mathbf{\Omega}^a}}(t) \end{bmatrix}.$$

$\mathbf{U}(t)$ and $\mathbf{V}(t)$ are $n_{\mathbf{\Omega}^a} \times n_{\mathbf{\Omega}^a}$ and $n_\mathbf{u} \times n_\mathbf{u}$ orthogonal matrix functions. Writing $\mathbf{V}(t)$ as $\begin{bmatrix} \mathbf{V}_c(t) & \mathbf{V}_s(t) \end{bmatrix}$, where $\mathbf{V}_c(t)$ and $\mathbf{V}_s(t)$ are $n_\mathbf{u} \times n_{\mathbf{\Omega}^a}$ and $n_\mathbf{u} \times (n_\mathbf{u} - n_{\mathbf{\Omega}^a})$ matrix functions gives

$$\mathbf{\Omega}_\mathbf{u}^a(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\theta}_0) = \mathbf{U}(t)\mathbf{\Sigma}_c(t)\mathbf{V}_c(t)^T, \quad (7)$$

along with the identities:

$$\begin{aligned} \mathbf{V}_c(t)^T\mathbf{V}_c(t) &= \mathbf{I}_{n_{\mathbf{\Omega}^a}},\\ \mathbf{V}_s(t)^T\mathbf{V}_s(t) &= \mathbf{I}_{(n_\mathbf{u} - n_{\mathbf{\Omega}^a})},\\ \mathbf{V}_c(t)^T\mathbf{V}_s(t) &= \mathbf{0}_{n_{\mathbf{\Omega}^a} \times (n_\mathbf{u} - n_{\mathbf{\Omega}^a})},\\ \mathbf{V}_s(t)^T\mathbf{V}_c(t) &= \mathbf{0}_{(n_\mathbf{u} - n_{\mathbf{\Omega}^a}) \times n_{\mathbf{\Omega}^a}}. \end{aligned} \quad (8)$$

Property 2 (Constraint- and Sensitivity-seeking Directions): The columns of the orthogonal matrices $\mathbf{V}_c(t)$ and $\mathbf{V}_s(t)$ span the subspaces of constraint- and sensitivity-seeking directions at time t, respectively.

Based on Property 2, input-variation functions $\boldsymbol{\xi}^\mathbf{u} : [t_0, t_\mathrm{f}] \to \mathbb{R}^{n_\mathbf{u}}$ can be split, at each time t, as:

$$\boldsymbol{\xi}^\mathbf{u}(t) = \mathbf{V}_c(t)\boldsymbol{\xi}_c^\mathbf{u}(t) + \mathbf{V}_s(t)\boldsymbol{\xi}_s^\mathbf{u}(t),$$

with $\boldsymbol{\xi}_c^\mathbf{u} : [t_0, t_\mathrm{f}] \to \mathbb{R}^{n_{\mathbf{\Omega}^a}}$ and $\boldsymbol{\xi}_s^\mathbf{u} : [t_0, t_\mathrm{f}] \to \mathbb{R}^{n_\mathbf{u} - n_{\mathbf{\Omega}^a}}$ standing for the projections of $\boldsymbol{\xi}^\mathbf{u}$ in the constraint- and sensitivity-seeking subspaces, respectively. Using (8), these latter functions are obtained as:

$$\begin{aligned} \boldsymbol{\xi}_c^\mathbf{u}(t) &= \mathbf{V}_c(t)^T\boldsymbol{\xi}^\mathbf{u}(t),\\ \boldsymbol{\xi}_s^\mathbf{u}(t) &= \mathbf{V}_s(t)^T\boldsymbol{\xi}^\mathbf{u}(t). \end{aligned} \quad (9)$$

IV. Variations in Constraint- and Sensitivity-seeking Directions

In this section, we derive expressions for the first-order variations in optimal inputs along both the constraint-seeking and sensitivity-seeking directions, as caused by small variations of the parameters $\boldsymbol{\theta}$ in the directions $\boldsymbol{\xi}^{\boldsymbol{\theta}}$.

Let us assume that, when the parameters change from $\boldsymbol{\theta}_0$ to $\tilde{\boldsymbol{\theta}}(\eta) := \boldsymbol{\theta}_0 + \eta\boldsymbol{\xi}^{\boldsymbol{\theta}}$, with $|\eta| \ll 1$, the optimal inputs change from $\mathbf{u}^*$ to $\tilde{\mathbf{u}}^*$, i.e., $\tilde{\mathbf{u}}^*$ is the optimal input for the perturbed problem. Let us denote by $\tilde{\mathbf{x}}^*$, $\tilde{\boldsymbol{\lambda}}^*$, and $\tilde{\boldsymbol{\mu}}^*$ the corresponding perturbed states, adjoints and multiplier functions.

The SOSC ensure that the solution of the optimal control problem is differentiable with respect to η at $\eta = 0$, a.e. in $[t_0, t_\mathrm{f}]$. Therefore, there exists $\delta > 0$ such that $\forall \eta \in \mathcal{B}_\delta(0)$, a first-order approximation of the 4-tuple $(\tilde{\mathbf{x}}^*(t;\eta), \tilde{\mathbf{u}}^*(t;\eta), \tilde{\boldsymbol{\lambda}}^*(t;\eta), \tilde{\boldsymbol{\mu}}^*(t;\eta))$ is obtained as:

$$\begin{aligned}
\tilde{\mathbf{x}}^*(t;\eta) &= \mathbf{x}^*(t) + \eta\boldsymbol{\xi}^{\mathbf{x}}(t) + O(\eta^2),\\
\tilde{\mathbf{u}}^*(t;\eta) &= \mathbf{u}^*(t) + \eta\boldsymbol{\xi}^{\mathbf{u}}(t) + O(\eta^2),\\
\tilde{\boldsymbol{\lambda}}^*(t;\eta) &= \boldsymbol{\lambda}^*(t) + \eta\boldsymbol{\xi}^{\boldsymbol{\lambda}}(t) + O(\eta^2),\\
\tilde{\boldsymbol{\mu}}^*(t;\eta) &= \boldsymbol{\mu}^*(t) + \eta\boldsymbol{\xi}^{\boldsymbol{\mu}}(t) + O(\eta^2).
\end{aligned} \tag{10}$$

Moreover, the vector functions $\boldsymbol{\xi}^{\mathbf{u}}$, $\boldsymbol{\xi}^{\mathbf{x}}$, $\boldsymbol{\xi}^{\boldsymbol{\lambda}}$ and $\boldsymbol{\xi}^{\boldsymbol{\mu}}$ obey the following linear two-point-boundary-value problem (TPBVP) a.e. in $[t_0, t_\mathrm{f}]$:[1]

$$\dot{\boldsymbol{\xi}}^{\mathbf{x}}(t) = \mathbf{f}_{\mathbf{x}}[t]\boldsymbol{\xi}^{\mathbf{x}}(t) + \mathbf{f}_{\mathbf{u}}[t]\boldsymbol{\xi}^{\mathbf{u}}(t) + \mathbf{f}_{\boldsymbol{\theta}}[t]\boldsymbol{\xi}^{\boldsymbol{\theta}}, \tag{11}$$

$$\boldsymbol{\xi}^{\mathbf{x}}(t_0) = \mathbf{h}_{\boldsymbol{\theta}}(\boldsymbol{\theta}_0)\boldsymbol{\xi}^{\boldsymbol{\theta}},$$

$$\begin{aligned}
\mathbf{0} = {} & \mathcal{H}_{\mathbf{ux}}[t]\boldsymbol{\xi}^{\mathbf{x}}(t) + \mathcal{H}_{\mathbf{uu}}[t]\boldsymbol{\xi}^{\mathbf{u}}(t) + \mathcal{H}_{\mathbf{u}\boldsymbol{\lambda}}[t]\boldsymbol{\xi}^{\boldsymbol{\lambda}}(t)\\
& + \mathcal{H}_{\mathbf{u}\boldsymbol{\mu}}[t]\boldsymbol{\xi}^{\boldsymbol{\mu}}(t) + \mathcal{H}_{\mathbf{u}\boldsymbol{\theta}}[t]\boldsymbol{\xi}^{\boldsymbol{\theta}},
\end{aligned} \tag{12}$$

$$\begin{aligned}
\dot{\boldsymbol{\xi}}^{\boldsymbol{\lambda}}(t) = {} & -\mathcal{H}_{\mathbf{xx}}[t]\boldsymbol{\xi}^{\mathbf{x}}(t) - \mathcal{H}_{\mathbf{xu}}[t]\boldsymbol{\xi}^{\mathbf{u}}(t)\\
& - \mathcal{H}_{\mathbf{x}\boldsymbol{\lambda}}[t]\boldsymbol{\xi}^{\boldsymbol{\lambda}}(t) - \mathcal{H}_{\mathbf{x}\boldsymbol{\mu}}[t]\boldsymbol{\xi}^{\boldsymbol{\mu}}(t) - \mathcal{H}_{\mathbf{x}\boldsymbol{\theta}}[t]\boldsymbol{\xi}^{\boldsymbol{\theta}},
\end{aligned} \tag{13}$$

$$\boldsymbol{\xi}^{\boldsymbol{\lambda}}(t_\mathrm{f}) = \psi_{\mathbf{xx}}[t_\mathrm{f}]\boldsymbol{\xi}^{\mathbf{x}}(t_\mathrm{f}) + \psi_{\mathbf{x}\boldsymbol{\theta}}[t_\mathrm{f}]\boldsymbol{\xi}^{\boldsymbol{\theta}},$$

$$\begin{aligned}
\mathbf{0} = {} & \mathrm{diag}(\boldsymbol{\Omega}[t])\boldsymbol{\xi}^{\boldsymbol{\mu}}(t) + \mathrm{diag}(\boldsymbol{\mu}^*(t))\left\{\boldsymbol{\Omega}_{\mathbf{x}}[t]^T\boldsymbol{\xi}^{\mathbf{x}}(t)\right.\\
& \left. + \boldsymbol{\Omega}_{\mathbf{u}}[t]^T\boldsymbol{\xi}^{\mathbf{u}}(t) + \boldsymbol{\Omega}_{\boldsymbol{\theta}}[t]^T\boldsymbol{\xi}^{\boldsymbol{\theta}}\right\}.
\end{aligned} \tag{14}$$

Note that the continuity of $\mathbf{x}^*$ and $\boldsymbol{\lambda}^*$ in $[t_0, t_\mathrm{f}]$ ensures that the variation functions $\boldsymbol{\xi}^{\mathbf{x}}$ and $\boldsymbol{\xi}^{\boldsymbol{\lambda}}$ are themselves continuous in $[t_0, t_\mathrm{f}]$, including at the switching times. On the other hand, the variation functions $\boldsymbol{\xi}^{\mathbf{u}}$ and $\boldsymbol{\xi}^{\boldsymbol{\mu}}$ may be discontinuous at the switching times; for the input variation ξ^{u_k}, for instance, the following relation holds at the switching time $t^*_{k_i}$:

$$\xi^{u_k}(t^{*+}_{k_i}) = \xi^{u_k}(t^{*-}_{k_i}) + \left[\dot{u}^*_k(t^{*+}_{k_i}) - \dot{u}^*_k(t^{*-}_{k_i})\right]\xi^{t_{k_i}}, \tag{15}$$

where $\xi^{t_{k_i}}$ stands for the variation in switching time (see Section V).

[1] For sake of notational simplicity, the following compact notation is used subsequently:

$$\begin{aligned}
y[t] &:= y(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\mu}^*(t), \boldsymbol{\theta}_0),\\
\tilde{y}[t] &:= y(\tilde{\mathbf{x}}^*(t), \tilde{\mathbf{u}}^*(t), \tilde{\boldsymbol{\lambda}}^*(t), \tilde{\boldsymbol{\mu}}^*(t), \tilde{\boldsymbol{\theta}}).
\end{aligned}$$

By combining (12) and (14), and after basic algebraic manipulations, one obtains:

$$\begin{bmatrix}\boldsymbol{\xi}^{\mathbf{u}}(t)\\ \boldsymbol{\xi}^{\boldsymbol{\mu}}(t)\end{bmatrix} = \begin{bmatrix}\mathcal{A}_1(t)\\ \mathcal{A}_2(t)\end{bmatrix}\boldsymbol{\xi}^{\boldsymbol{\lambda}}(t) + \begin{bmatrix}\mathcal{B}_1(t)\\ \mathcal{B}_2(t)\end{bmatrix}\boldsymbol{\xi}^{\mathbf{x}}(t) + \begin{bmatrix}\mathcal{C}_1(t)\\ \mathcal{C}_2(t)\end{bmatrix}\boldsymbol{\xi}^{\boldsymbol{\theta}}, \tag{16}$$

where

$$\begin{aligned}
\begin{bmatrix}\mathcal{A}_1(t)\\ \mathcal{A}_2(t)\end{bmatrix} &:= -\mathcal{M}(t)^{-1}\begin{bmatrix}\mathcal{H}_{\mathbf{u}\boldsymbol{\lambda}}[t]\\ \mathbf{0}\end{bmatrix},\\
\begin{bmatrix}\mathcal{B}_1(t)\\ \mathcal{B}_2(t)\end{bmatrix} &:= -\mathcal{M}(t)^{-1}\begin{bmatrix}\mathcal{H}_{\mathbf{ux}}[t]\\ \mathrm{diag}(\boldsymbol{\mu}^*(t))\boldsymbol{\Omega}_{\mathbf{x}}[t]^T\end{bmatrix},\\
\begin{bmatrix}\mathcal{C}_1(t)\\ \mathcal{C}_2(t)\end{bmatrix} &:= -\mathcal{M}(t)^{-1}\begin{bmatrix}\mathcal{H}_{\mathbf{u}\boldsymbol{\theta}}[t]\\ \mathrm{diag}(\boldsymbol{\mu}^*(t))\boldsymbol{\Omega}_{\boldsymbol{\theta}}[t]^T\end{bmatrix},
\end{aligned}$$

and

$$\mathcal{M}(t) := \begin{bmatrix}\mathcal{H}_{\mathbf{uu}}[t] & \mathcal{H}_{\mathbf{u}\boldsymbol{\mu}}[t]\\ \mathrm{diag}(\boldsymbol{\mu}^*(t))\boldsymbol{\Omega}_{\mathbf{u}}[t]^T & \mathrm{diag}(\boldsymbol{\Omega}[t])\end{bmatrix}.$$

In particular, $\mathcal{M}(t)$ is known to be nonsingular a.e. in $[t_0, t_\mathrm{f}]$ from the SOSC.

Next, the generalized backward sweep method is applied to eliminate $\boldsymbol{\xi}^{\boldsymbol{\lambda}}$ from (16) [6]. By setting

$$\boldsymbol{\xi}^{\boldsymbol{\lambda}}(t) = \mathbf{S}^{\mathbf{x}}(t)\boldsymbol{\xi}^{\mathbf{x}}(t) + \mathbf{S}^{\boldsymbol{\theta}}(t)\boldsymbol{\xi}^{\boldsymbol{\theta}}, \tag{17}$$

and substituting it in (13), it can be shown, after appropriate rearrangements using (16), that $\mathbf{S}^{\mathbf{x}}(t)$ and $\mathbf{S}^{\boldsymbol{\theta}}(t)$, called the *sweep matrices*, must satisfy the following Riccati differential equations:

$$\begin{aligned}
\dot{\mathbf{S}}^{\mathbf{x}}(t) = {} & -\mathbf{S}^{\mathbf{x}}(t)\mathcal{H}_{\mathbf{x}\boldsymbol{\lambda}}[t] - \mathcal{H}_{\mathbf{x}\boldsymbol{\lambda}}[t]^T\mathbf{S}^{\mathbf{x}}(t) - \mathcal{H}_{\mathbf{xx}}[t]\\
& - \left\{\mathcal{H}_{\mathbf{xu}}[t] + \mathbf{S}^{\mathbf{x}}(t)\mathcal{H}_{\boldsymbol{\lambda}\mathbf{u}}[t]\right\}\mathbf{K}^{\mathbf{x}}(t),\\
\mathbf{S}^{\mathbf{x}}(t_\mathrm{f}) = {} & \psi_{\mathbf{xx}}(t_\mathrm{f}),
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
\dot{\mathbf{S}}^{\boldsymbol{\theta}}(t) = {} & -\mathbf{S}^{\mathbf{x}}(t)\mathcal{H}_{\boldsymbol{\theta}\boldsymbol{\lambda}}[t] - \mathcal{H}_{\mathbf{x}\boldsymbol{\lambda}}[t]^T\mathbf{S}^{\boldsymbol{\theta}}(t) - \mathcal{H}_{\mathbf{x}\boldsymbol{\theta}}[t]\\
& - \left\{\mathcal{H}_{\mathbf{xu}}[t] + \mathbf{S}^{\boldsymbol{\theta}}(t)\mathcal{H}_{\boldsymbol{\lambda}\mathbf{u}}[t]\right\}\mathbf{K}^{\boldsymbol{\theta}}(t),\\
\mathbf{S}^{\boldsymbol{\theta}}(t_\mathrm{f}) = {} & \psi_{\mathbf{x}\boldsymbol{\theta}}(t_\mathrm{f}),
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
\mathbf{K}^{\mathbf{x}}(t) &:= \begin{bmatrix}\mathbf{K}^{\mathbf{x}}_1(t)\\ \mathbf{K}^{\mathbf{x}}_2(t)\end{bmatrix} = \begin{bmatrix}\mathcal{A}_1(t)\\ \mathcal{A}_2(t)\end{bmatrix}\mathbf{S}^{\mathbf{x}}(t) + \begin{bmatrix}\mathcal{B}_1(t)\\ \mathcal{B}_2(t)\end{bmatrix},\\
\mathbf{K}^{\boldsymbol{\theta}}(t) &:= \begin{bmatrix}\mathbf{K}^{\boldsymbol{\theta}}_1(t)\\ \mathbf{K}^{\boldsymbol{\theta}}_2(t)\end{bmatrix} = \begin{bmatrix}\mathcal{A}_1(t)\\ \mathcal{A}_2(t)\end{bmatrix}\mathbf{S}^{\boldsymbol{\theta}}(t) + \begin{bmatrix}\mathcal{C}_1(t)\\ \mathcal{C}_2(t)\end{bmatrix}.
\end{aligned} \tag{20}$$

Using (17) in (16) gives:

$$\begin{aligned}
\boldsymbol{\xi}^{\mathbf{u}}(t) &= \mathbf{K}^{\mathbf{x}}_1(t)\boldsymbol{\xi}^{\mathbf{x}}(t) + \mathbf{K}^{\boldsymbol{\theta}}_1(t)\boldsymbol{\xi}^{\boldsymbol{\theta}},\\
\boldsymbol{\xi}^{\boldsymbol{\mu}}(t) &= \mathbf{K}^{\mathbf{x}}_2(t)\boldsymbol{\xi}^{\mathbf{x}}(t) + \mathbf{K}^{\boldsymbol{\theta}}_2(t)\boldsymbol{\xi}^{\boldsymbol{\theta}}.
\end{aligned} \tag{21}$$

In turn, $\boldsymbol{\xi}^{\mathbf{u}}(t)$ can be substituted in (11) to compute $\boldsymbol{\xi}^{\mathbf{x}}(t)$ as a function of $\boldsymbol{\xi}^{\boldsymbol{\theta}}$ only:

$$\begin{aligned}
\dot{\boldsymbol{\xi}}^{\mathbf{x}}(t) = {} & \left\{\mathbf{f}_{\mathbf{x}}[t] + \mathbf{f}_{\mathbf{u}}[t]\mathbf{K}^{\mathbf{x}}_1(t)\right\}\boldsymbol{\xi}^{\mathbf{x}}(t)\\
& + \left\{\mathbf{f}_{\boldsymbol{\theta}}[t] + \mathbf{f}_{\mathbf{u}}[t]\mathbf{K}^{\boldsymbol{\theta}}_1(t)\right\}\boldsymbol{\xi}^{\boldsymbol{\theta}},
\end{aligned}$$

with the initial conditions $\boldsymbol{\xi}^{\mathbf{x}}(t_0) = \mathbf{h}_{\boldsymbol{\theta}}(\boldsymbol{\theta}_0)$. The solution of this inhomogeneous linear differential equation can be written as

$$\boldsymbol{\xi}^{\mathbf{x}}(t) = \boldsymbol{\Xi}(t)\boldsymbol{\xi}^{\boldsymbol{\theta}}, \tag{22}$$

where

$$\begin{aligned}\boldsymbol{\Xi}(t) := &\boldsymbol{\Phi}(t-t_0)\mathbf{h}_{\boldsymbol{\theta}}(\boldsymbol{\theta}_0)\\ &+ \int_{t_0}^{t} \boldsymbol{\Phi}(t-\tau)\left\{\mathbf{f}_{\boldsymbol{\theta}}[\tau] + \mathbf{f}_{\mathbf{u}}[\tau]\mathbf{K}_1^{\boldsymbol{\theta}}(\tau)\right\} d\tau,\end{aligned}$$

and $\boldsymbol{\Phi}(t)$ satisfies

$$\dot{\boldsymbol{\Phi}}(t) = \left\{\mathbf{f}_{\mathbf{x}}[t] + \mathbf{f}_{\mathbf{u}}[t]\mathbf{K}_1^{\mathbf{x}}(t)\right\}\boldsymbol{\Phi}(t); \quad \boldsymbol{\Phi}(0) = \mathbf{I}_{n_{\mathbf{x}}}. \tag{23}$$

In summary, the expressions linking the first-order variations in inputs, states, adjoints and multipliers to the parameter variations can be written collectively as:

$$\begin{bmatrix}\boldsymbol{\xi}^{\mathbf{x}}(t)\\ \boldsymbol{\xi}^{\mathbf{u}}(t)\\ \boldsymbol{\xi}^{\boldsymbol{\lambda}}(t)\\ \boldsymbol{\xi}^{\boldsymbol{\mu}}(t)\end{bmatrix} = \left(\begin{bmatrix}\mathbf{I}_{n_{\mathbf{x}}}\\ \mathbf{K}_1^{\mathbf{x}}(t)\\ \mathbf{S}^{\mathbf{x}}(t)\\ \mathbf{K}_2^{\mathbf{x}}(t)\end{bmatrix}\boldsymbol{\Xi}(t) + \begin{bmatrix}\mathbf{0}_{n_{\mathbf{x}}\times n_{\boldsymbol{\theta}}}\\ \mathbf{K}_1^{\boldsymbol{\theta}}(t)\\ \mathbf{S}^{\boldsymbol{\theta}}(t)\\ \mathbf{K}_2^{\boldsymbol{\theta}}(t)\end{bmatrix}\right)\boldsymbol{\xi}^{\boldsymbol{\theta}}. \tag{24}$$

Next, we need to express the variations $\boldsymbol{\xi}^{\mathbf{u}}(t)$ in terms of variations along the constraint- and sensitivity-seeking directions.

• *Input Variation $\boldsymbol{\xi}_c^{\mathbf{u}}$ along Constraint-seeking Directions:* The path constraints must be satisfied by both the nominal and perturbed optimal inputs. Hence, the first variation of the path constraints perturbed in the direction $(\boldsymbol{\xi}^{\mathbf{x}}(t), \boldsymbol{\xi}^{\mathbf{u}}(t), \boldsymbol{\xi}^{\boldsymbol{\theta}})$ at $(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\theta}_0)$ must be zero:

$$\mathbf{0} = \left.\frac{\partial}{\partial\eta}\tilde{\boldsymbol{\Omega}}^a[t]\right|_{\eta=0} = \boldsymbol{\Omega}_{\mathbf{x}}^a[t]\boldsymbol{\xi}^{\mathbf{x}}(t) + \boldsymbol{\Omega}_{\mathbf{u}}^a[t]\boldsymbol{\xi}^{\mathbf{u}}(t) + \boldsymbol{\Omega}_{\boldsymbol{\theta}}^a[t]\boldsymbol{\xi}^{\boldsymbol{\theta}}.$$

Using (7), (9) and (24), this equation can be rearranged as

$$\begin{aligned}\left\{\boldsymbol{\Omega}_{\mathbf{x}}^a[t]\boldsymbol{\Xi}(t) + \boldsymbol{\Omega}_{\boldsymbol{\theta}}^a[t]\right\}\boldsymbol{\xi}^{\boldsymbol{\theta}} &= -\mathbf{U}(t)\boldsymbol{\Sigma}_c(t)\mathbf{V}_c(t)^T\boldsymbol{\xi}^{\mathbf{u}}(t),\\ &= -\mathbf{U}(t)\boldsymbol{\Sigma}_c(t)\boldsymbol{\xi}_c^{\mathbf{u}}(t).\end{aligned}$$

From (7), it is seen that $\mathbf{U}(t)\boldsymbol{\Sigma}_c(t) = \boldsymbol{\Omega}_{\mathbf{u}}^a[t]\mathbf{V}_c(t)$ and $\boldsymbol{\Omega}_{\mathbf{u}}^a[t]\mathbf{V}_c(t)$ is always invertible. Therefore,

$$\boldsymbol{\xi}_c^{\mathbf{u}}(t) = -\left\{\boldsymbol{\Omega}_{\mathbf{u}}^a[t]\mathbf{V}_c(t)\right\}^{-1}\left\{\boldsymbol{\Omega}_{\mathbf{x}}^a[t]\boldsymbol{\Xi}(t) + \boldsymbol{\Omega}_{\boldsymbol{\theta}}^a[t]\right\}\boldsymbol{\xi}^{\boldsymbol{\theta}}. \tag{25}$$

• *Input Variation $\boldsymbol{\xi}_s^{\mathbf{u}}$ along Sensitivity-seeking Directions:* Both the nominal and perturbed optimal inputs must also satisfy the optimality conditions (4), i.e., the first variation of $\mathcal{H}_{\mathbf{u}}$ at $(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\mu}^*(t), \boldsymbol{\theta}_0)$ in the direction $(\boldsymbol{\xi}^{\mathbf{x}}(t), \boldsymbol{\xi}^{\mathbf{u}}(t), \boldsymbol{\xi}^{\boldsymbol{\lambda}}(t), \boldsymbol{\xi}^{\boldsymbol{\mu}}(t), \boldsymbol{\xi}^{\boldsymbol{\theta}})$ must be zero:

$$\begin{aligned}\mathbf{0} = &\left.\frac{\partial}{\partial\eta}\tilde{\mathcal{H}}_{\mathbf{u}}[t]\right|_{\eta=0}\\ =&\mathcal{H}_{\mathbf{ux}}[t]\boldsymbol{\xi}^{\mathbf{x}}(t) + \mathcal{H}_{\mathbf{uu}}[t]\boldsymbol{\xi}^{\mathbf{u}}(t) + \mathcal{H}_{\mathbf{u}\boldsymbol{\lambda}}[t]\boldsymbol{\xi}^{\boldsymbol{\lambda}}(t)\\ &+ \mathcal{H}_{\mathbf{u}\boldsymbol{\mu}}[t]\boldsymbol{\xi}^{\boldsymbol{\mu}}(t) + \mathcal{H}_{\mathbf{u}\boldsymbol{\theta}}[t]\boldsymbol{\xi}^{\boldsymbol{\theta}}.\end{aligned}$$

Substituting the values of $\boldsymbol{\xi}^{\mathbf{x}}(t)$, $\boldsymbol{\xi}^{\boldsymbol{\lambda}}(t)$ and $\boldsymbol{\xi}^{\boldsymbol{\mu}}(t)$ from (24) and rearranging gives

$$\boldsymbol{\xi}^{\mathbf{u}}(t) = -\mathcal{H}_{\mathbf{uu}}[t]^{-1}\left\{\mathcal{Z}^{\mathbf{x}}(t)\boldsymbol{\Xi}(t) + \mathcal{Z}^{\boldsymbol{\theta}}(t)\right\}\boldsymbol{\xi}^{\boldsymbol{\theta}},$$

where

$$\begin{aligned}\mathcal{Z}^{\mathbf{x}}(t) &:= \mathcal{H}_{\mathbf{ux}}[t] + \mathcal{H}_{\mathbf{u}\boldsymbol{\lambda}}[t]\mathbf{S}^{\mathbf{x}}(t) + \mathcal{H}_{\mathbf{u}\boldsymbol{\mu}}[t]\mathbf{K}_2^{\mathbf{x}}(t),\\ \mathcal{Z}^{\boldsymbol{\theta}}(t) &:= \mathcal{H}_{\mathbf{u}\boldsymbol{\theta}}[t] + \mathcal{H}_{\mathbf{u}\boldsymbol{\lambda}}[t]\mathbf{S}^{\boldsymbol{\theta}}(t) + \mathcal{H}_{\mathbf{u}\boldsymbol{\mu}}[t]\mathbf{K}_2^{\boldsymbol{\theta}}(t).\end{aligned}$$

The nonsingularity of $\mathcal{H}_{\mathbf{uu}}$ along the nominal solution follows directly from the SOSC (strict Legendre-Clebsch condition). Finally, using (9), one obtains:

$$\boldsymbol{\xi}_s^{\mathbf{u}}(t) = -\mathbf{V}_s(t)^T\mathcal{H}_{\mathbf{uu}}[t]^{-1}\left\{\mathcal{Z}^{\mathbf{x}}(t)\boldsymbol{\Xi}(t) + \mathcal{Z}^{\boldsymbol{\theta}}(t)\right\}\boldsymbol{\xi}^{\boldsymbol{\theta}}. \tag{26}$$

V. Variations of Switching Times

Consider the switching time $t_{k_i}^*$ associated with the constraint Ω_i in the nominal optimal solution. Without loss of generality, suppose that $\Omega_i[t_{k_i}^{*-}] < 0$ and $\Omega_i[t_{k_i}^{*+}] = 0$, i.e., $t_{k_i}^*$ is an entry time for that constraint. From the differentiability of the optimal solution with respect to η at $\eta = 0$ (SOSC), there exists $\delta > 0$ such that, $\forall \eta \in \mathcal{B}_\delta(0)$, a first-order approximation of the switching time for the perturbed optimal system $\tilde{t}_{k_i}^*$ is as follows:

$$\tilde{t}_{k_i}^*(\eta) = t_{k_i}^* + \eta\xi^{t_{k_i}} + o(\eta). \tag{27}$$

Following the ideas in [7], define $\mathbf{u}_{uc}$ as the *unconstrained inputs* obtained by extrapolating the input trajectories beyond the switching instant $t_{k_i}^*$ in such a way that they still satisfy the same optimality conditions as at $t_{k_i}^{*-}$. Let the state trajectories corresponding to $\mathbf{u}_{uc}$ be denoted by $\mathbf{x}_{uc}$, and define the functions ω_i and $\tilde{\omega}_i$ as

$$\begin{aligned}\omega_i[t] &:= \Omega_i(t, \mathbf{x}_{uc}^*(t), \mathbf{u}_{uc}^*(t), \boldsymbol{\theta}_0),\\ \tilde{\omega}_i[t] &:= \Omega_i(t, \tilde{\mathbf{x}}_{uc}^*(t;\eta), \tilde{\mathbf{u}}_{uc}^*(t;\eta), \tilde{\boldsymbol{\theta}}(\eta)).\end{aligned}$$

Note that $\omega_i[t_{k_i}^{*+}] = 0$ and $\tilde{\omega}_i[t_{k_i}^{*+} + \eta\xi^{t_{k_i}}] = 0$, $\forall \eta \in \mathcal{B}_\delta(0)$, which leads to

$$\begin{aligned}0 = &\left.\frac{\partial}{\partial\eta}\tilde{\omega}_i[t_{k_i}^{*+}]\right|_{\eta=0}\\ =&(\omega_i)_t[t_{k_i}^{*+}]\xi^{t_{k_i}} + (\omega_i)_{\mathbf{x}}[t_{k_i}^{*+}]\left\{\xi_{uc}^{\mathbf{x}}(t_{k_i}^{*+}) + \dot{\mathbf{x}}(t_{k_i}^{*+})\xi^{t_{k_i}}\right\}\\ &+ (\omega_i)_{\mathbf{u}}[t_{k_i}^{*+}]\left\{\xi_{uc}^{\mathbf{u}}(t_{k_i}^{*+}) + \dot{\mathbf{u}}(t_{k_i}^{*+})\xi^{t_{k_i}}\right\} + (\omega_i)_{\boldsymbol{\theta}}[t_{k_i}^{*+}]\xi^{\boldsymbol{\theta}},\\ =&\left\{(\omega_i)_t + (\omega_i)_{\mathbf{x}}\dot{\mathbf{x}}_{uc} + (\omega_i)_{\mathbf{u}}\dot{\mathbf{u}}_{uc}\right\}_{t_{k_i}^{*+}}\xi^{t_{k_i}}\\ &+ \left\{(\omega_i)_{\mathbf{x}}\xi_{uc}^{\mathbf{x}} + (\omega_i)_{\mathbf{u}}\xi_{uc}^{\mathbf{u}} + (\omega_i)_{\boldsymbol{\theta}}\xi^{\boldsymbol{\theta}}\right\}_{t_{k_i}^{*+}}.\end{aligned}$$

Hence, assuming, as in [7], that $\dot{\omega}_i$ is non-zero at $t_{k_i}^{*+}$ gives:

$$\xi^{t_{k_i}} = -\frac{1}{\dot{\omega}_i}\left\{(\omega_i)_{\mathbf{x}}\xi_{uc}^{\mathbf{x}} + (\omega_i)_{\mathbf{u}}\xi_{uc}^{\mathbf{u}} + (\omega_i)_{\boldsymbol{\theta}}\xi^{\boldsymbol{\theta}}\right\}_{t_{k_i}^{*+}}.$$

Since, by construction, the unconstrained inputs $\mathbf{u}_{uc}$ are continuous across the switching time $t_{k_i}^*$, so are the response trajectories $\mathbf{x}_{uc}$, and the corresponding variations are themselves continuous:

$$\xi_{uc}^{\mathbf{u}}(t_{k_i}^{*+}) = \xi_{uc}^{\mathbf{u}}(t_{k_i}^{*-}), \text{ and } \quad \xi_{uc}^{\mathbf{x}}(t_{k_i}^{*+}) = \xi_{uc}^{\mathbf{x}}(t_{k_i}^{*-}).$$

Moreover, the unconstrained inputs being the same as the nominal optimal inputs up to $t_{k_i}^{*-}$, one can write:

$$\xi_{uc}^{\mathbf{u}}(t_{k_i}^{*-}) = \xi^{\mathbf{u}}(t_{k_i}^{*-}), \text{ and } \quad \xi_{uc}^{\mathbf{x}}(t_{k_i}^{*-}) = \xi^{\mathbf{x}}(t_{k_i}^{*-}),$$

that is,

$$\xi^{t_{k_i}} = -\frac{1}{\dot{\omega}_i} \left\{ (\omega_i)_{\mathbf{x}} \xi^{\mathbf{x}} + (\omega_i)_{\mathbf{u}} \xi^{\mathbf{u}} + (\omega_i)_{\boldsymbol{\theta}} \xi^{\boldsymbol{\theta}} \right\}_{t^{*-}_{k_i}}.$$

Finally, expressing $\xi^{\mathbf{x}}(t^{*-}_{k_i})$ and $\xi^{\mathbf{u}}(t^{*-}_{k_i})$ from (24) gives

$$\xi^{t_{k_i}} = -\frac{1}{\dot{\Omega}_i[t^{*-}_{k_i}]} \left\{ \mathcal{G}^{\mathbf{x}}(t)\boldsymbol{\Xi}(t) + \mathcal{G}^{\boldsymbol{\theta}}(t) \right\}_{t^{*-}_{k_i}} \boldsymbol{\xi}^{\boldsymbol{\theta}}, \qquad (28)$$

where

$$\mathcal{G}^{\mathbf{x}}(t) := (\Omega_i)_{\mathbf{x}}[t] + (\Omega_i)_{\mathbf{u}}[t]\mathbf{K}^{\mathbf{x}}_1(t),$$
$$\mathcal{G}^{\boldsymbol{\theta}}(t) := (\Omega_i)_{\boldsymbol{\theta}}[t] + (\Omega_i)_{\mathbf{u}}[t]\mathbf{K}^{\boldsymbol{\theta}}_1(t).$$

Notice the analogy among (25), (26) and (28).

VI. Illustrative Example

Consider the following parametric optimal control problem with one input variable and one path constraint:

$$\begin{aligned}
\min_{u} \quad & \int_0^1 \left(x_1^2(t) + x_2^2(t) + 0.005u^2(t)\right) dt, \qquad (29)\\
\text{s.t.} \quad & \dot{x}_1(t) = x_2(t); \quad x_1(0) = 0,\\
& \dot{x}_2(t) = -x_2(t) + \theta\ u(t); \quad x_2(0) = -0.2,\\
& x_2(t) - u(t) + 0.5 - 8(t-0.5)^2 \le 0, \quad 0 \le t \le 1,
\end{aligned}$$

where θ stands for the uncertain system parameter, with nominal value $\theta_0 = 1$.

This problem is first solved using a direct sequential approach [8] with a piecewise-affine input parameterization on 90 non-equal time stages. The nominal optimal input u^* for Problem (29) consists of 3 arcs: an interior arc, followed by a boundary arc, and finally another interior arc. The optimal input is depicted in the top plot of Figure 1, while the path constraint is shown in the bottom plot. The corresponding optimal cost value and switching times t_1^* and t_2^* are reported in Table I.

TABLE I

Optimal nominal solution: cost value and switching times.

	Cost	t_1^*	t_2^*
$\theta_0 = 1$	$6.628\ 10^{-3}$	0.157	0.880

Regarding the methodology presented in Sections IV and V, one can calculate the variation functions ξ^u, $\boldsymbol{\xi}^{\mathbf{x}}$, $\boldsymbol{\xi}^{\boldsymbol{\lambda}}$ and ξ^μ following a variation of the system parameter θ such that $\eta\xi^\theta = 1$. The input variation $\xi^u(t)$ can be split into the contributions $\xi^u_c(t)$ and $\xi^u_s(t)$ (Figure 2). With a single input and the constraint active only in $[t_1^*, t_2^*]$, the functions $\xi^u_c(t)$ and $\xi^u_s(t)$ are not defined for all t in $[t_0, t_f]$. Observe that the variation is much larger in the sensitivity-seeking direction than in the constraint-seeking direction. The discontinuity in input variation at t_1^* and t_2^* is a consequence of the discontinuity in the time derivative of the optimal input — see (15) and Figure 1. Moreover, the variations of the switching times t_1^* and t_2^* are computed as:

$$\xi^{t_1} \approx -0.1090, \quad \text{and} \qquad \xi^{t_2} \approx 0.0567,$$

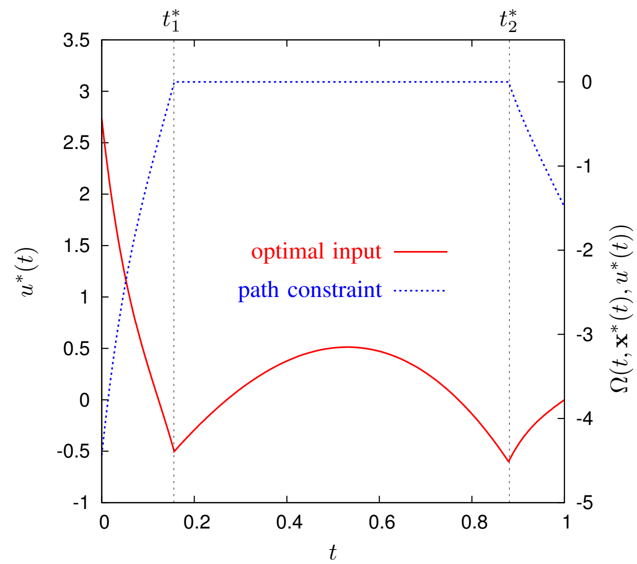

Fig. 1. Optimal nominal solution: optimal input (red solid line) and path constraint (blue dotted line).

an indication that the switching time t_1^* decreases, while t_2^* increases for $0 < \eta \ll 1$.

To illustrate the effect of selective input adaptation, a scenario is considered next wherein the system parameter θ is decreased by 25% (i.e., $\eta\xi^\theta := -0.25$). A strategy that selectively adapts the input profile along the sensitivity seeking direction as well as the switching times is shown in Figure 3; that is, no adaptation is made in the constraint seeking directions. Also shown in this figure are the nominal input profile along with the optimal input profile for the perturbed system for $\tilde{\theta} = 0.75$.

The lower plot in Figure 3 shows that the path constraint is satisfied in each case. In terms of performance, it is found

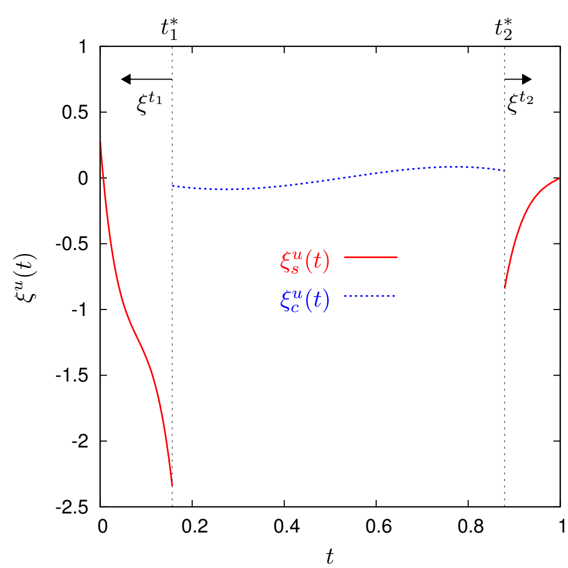

Fig. 2. Variation of the optimal input for $\eta\xi^\theta = 1$: input variation along the constraint-seeking directions (dashed line) and the sensitivity-seeking directions (solid line).

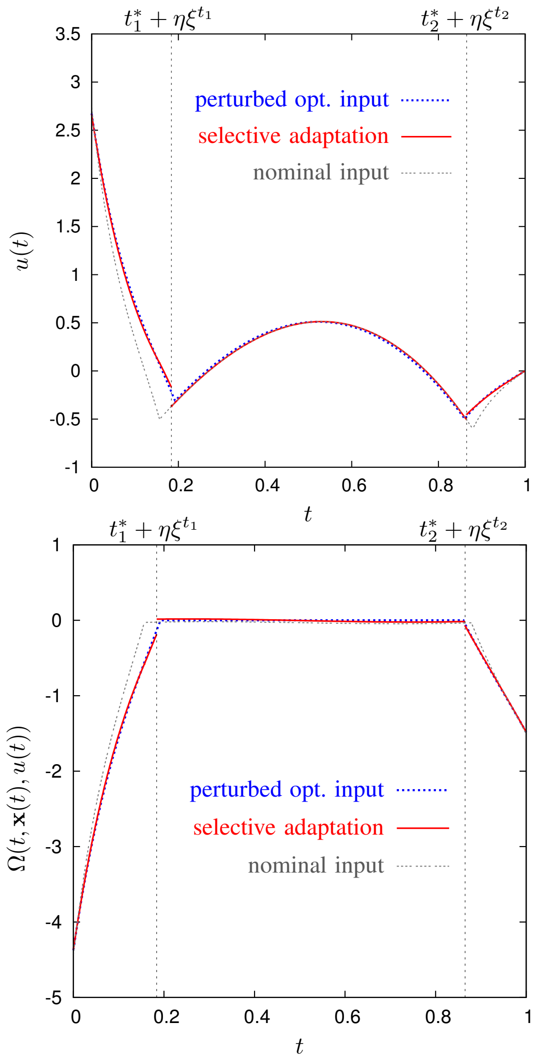

Fig. 3. Comparison of results of selective input adpatation with those of perturbed and nominal optimal systems: input profile (top plot) and path constraint (bottom plot).

that the selective adaptation strategy yields a cost of 5.910×10^{-3}, which is fairly close to the optimal cost for perturbed system, which is 5.836×10^{-3}, in comparison to the nominal cost of 7.534×10^{-3}. Accordingly, much of the optimality loss incurred by the parametric variation can be recovered without the need to adapt the input profile along constraint-seeking directions.

VII. Conclusions

Plant variations cause changes in the solution of optimal control problems. This paper has considered parametric variations and quantified these changes under the assumption of SOSC.

When the number of active constraints at a given time is less than the number of inputs, there exist input directions along which an infinitesimal step from the nominal values does not modify the active constraints. These directions have been labeled sensitivity-seeking directions. The remaining directions in $\mathbb{R}^{n_\mathbf{u}}$ that are orthogonal to the set of sensitivity-seeking directions have been labeled constraint-seeking directions. Orthogonal bases for the subspaces spanned by both sets of directions at a given time can be obtained from the singular value decomposition of the Jacobian matrix $\mathbf{\Omega}^a_\mathbf{u}(t)$.

The SOSC ensure that the optimal solution is differentiable, thus leading to a first-order approximation of the perturbed solution in its neighborhood. Based on this approximation, the input variations along the constraint- and sensitivity-seeking directions can be quantified. These input variations are found to be proportional to the parameter variations.

The third element of change in optimal inputs corresponds to variations of the switching times between arcs. By extrapolating the optimal inputs beyond the switching times so that they still satisfy the same optimality conditions as before switching, an expression for the variations in switching times has been obtained. These variations are also found to be proportional to the parameter variations.

Future directions of research will be to analyze how selective adaptation strategies translate in terms of performance improvement. To this end, the expressions derived in this paper for input variations along the constraint- and sensitivity-seeking directions as well as for variations in switching times will be particularly useful in quantifying respective cost variations. By making additional analysis, the same ideas will also be extended to singular optimal control problems as well as to problems involving terminal constraints. It is foreseen that such analysis will help develop preferential adaptation schemes for these various types of optimal control problems.

Acknowledgments

Financial support from the Swiss National Science Foundation (SNSF) under Grant #200020-113292 is gratefully acknowledged.

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