Accuracy Aspects of Iterative Correlation-Based Controller Tuning

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***Abstract*— The recently-proposed method for iterative correlation-based controller tuning that uses instrumental variables is considered. A confidence interval based on the covariance of the criterion function is introduced. An explicit expression for the variance of the controller parameter estimates around the optimal solution is developed and used to construct the optimal instruments. This expression in turn allows constructing a region around the obtained controller that contains the optimal controller with a certain probability level. The accuracy of the corresponding controller transfer function is also investigated. It is shown that the variance of the controller transfer function is proportional to the noise spectrum at the controller output and inversely proportional to the control error spectrum, the factor of proportionality being the ratio of the controller order to the number of data points. This result, which is asymptotic in both the controller order and the data length, is independent of the instrumental variables used.**

***Index Terms*— Controller tuning, correlation-based tuning, asymptotic variance analysis, instrumental variables.**

I. INTRODUCTION

Acquisition of process knowledge and its efficient use for control design are essential tasks of a control engineer. The intrinsic nonlinearities of industrial processes and the various complex behaviors they exhibit can turn the plant modeling/identification into a very challenging task. An alternative to model-based control design is to use the information collected on the plant *directly* for controller update. After numerous investigations in the framework of direct adaptive control [1], the so-called data-driven methods started to gain in popularity in the late 1990s when several methods appeared such as controller unfalsification [12], simultaneous perturbation stochastic approximation control [14], iterative feedback tuning [5] and virtual reference feedback tuning [2]. One of the main questions that arise in this research area is how to cope with the noise that necessarily corrupts the measurements and therefore also affects the closed-loop performance.

In the recently-proposed correlation-based approach to model following, the problem of measurement noise is addressed differently [7]. The underlying idea is inspired from the correlation approach that uses instrumental variables and is well known in the system identification community [13]. The controller parameters are tuned to make the closed-loop output error between the designed closed-loop system and the achieved one uncorrelated with the external reference signal. This way, the closed-loop output error ideally only contains the contribution of the noise, while the achieved closed-loop system captures the dynamics of the designed one. Moreover, the calculated controller parameters are not asymptotically affected by the noise. In [6], the tuning objective is reformulated as the minimization of the 2-norm of the correlation function between the closed-loop output error and the reference signal. A generalized correlation criterion that allows dealing with mixed sensitivity specifications is proposed in [11]. In [10], an adaptation of this approach to the disturbance rejection problem is considered.

In practice, the iterative solution of correlation equations may necessitate a large number of experiments for convergence. Furthermore, the limited amount of data points in each experiment affects the stochastic properties of the controller parameter estimates. In addition, process disturbances and measurement noise introduce errors in the solution. Therefore, for all iterative data-driven methods, it is of particular interest to study how fast the computed controllers approach the optimal one, as this is done in [4], [3]. Other important questions that arise with iterative methods is when to stop the iterations and how close the obtained controller is to the optimal one. In this paper, a confidence interval based on the covariance of the criterion function is introduced. This confidence interval helps determine to what extent the current controller decorrelates the closed-loop output error from the reference signal. An asymptotic expression for the accuracy of the controller parameters around the optimal solution is derived. This allows constructing a region around the obtained controller that contains the optimal controller with a certain probability level. An asymptotic expression for the accuracy of the controller transfer function estimate that characterizes this region is derived as well. Another reason for studying these properties is that the covariance matrix of the controller parameter estimates helps choose optimal instruments in the sense that they provide maximal accuracy.

The remainder of the paper is organized as follows. Some notations and preliminary facts about the correlation-based tuning approach are given in Section II. The accuracy aspects of this approach are discussed in Section III. Finally, some concluding remarks are given in Section IV.

II. Preliminaries

Let the output of some unknown true plant be described by the discrete-time model:

$$y(t) = G(q^{-1})u(t) \tag{1}$$

where q^{-1} is the backward-shift operator, $u(t)$ the input signal to the plant, and $G(q^{-1})$ a discrete-time transfer operator defined as:

$$G(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})} \tag{2}$$

The plant is controlled by the following one-degree-of-freedom controller:

$$K(q^{-1}, \rho) = \frac{S(q^{-1}, \rho)}{R(q^{-1}, \rho)} \tag{3}$$

where

$$R(q^{-1}, \rho) = 1 + r_1 q^{-1} + \cdots + r_n q^{-n} \tag{4}$$

$$S(q^{-1}, \rho) = s_0 + s_1 q^{-1} + \cdots + s_{n-1} q^{-n+1} \tag{5}$$

The controller parameter vector ρ is written as follows:

$$\rho^T = [{\rho^{(1)}}^T, {\rho^{(2)}}^T, \ldots, {\rho^{(n)}}^T] \tag{6}$$

where ${\rho^{(l)}}^T = [r_l, s_{l-1}]$, $l = 1, \ldots n$; $\dim(\rho) = n_\rho = 2n$. Denote

$$\varphi^T(\rho, t-1) = [-u(\rho, t-1), e(\rho, t)] \tag{7}$$

with $e(\rho, t) = r(t) - y(\rho, t)$, where $y(\rho, t)$ is the output of the achieved closed-loop system (Fig. 1), and $r(t)$ is the reference signal with spectrum $\Phi_r(\omega)$. Form the $2n$-dimensional vector

$$\phi^T(\rho, t) = [\varphi^T(\rho, t-1), \cdots, \varphi^T(\rho, t-n)]. \tag{8}$$

To facilitate the calculations and the discussion in the sequel, it is assumed, without any loss of generality, that a zero-mean weakly stationary random process $v(t)$ acts at the plant input (see upper part of Fig.1):

$$u(\rho, t) = K(q^{-1}, \rho)e(\rho, t) + v(t) \tag{9}$$

It is assumed that the measurements of $r(t)$ and $y(\rho, t)$ are available. The excitation signal $r(t)$ is assumed to be uncorrelated with the disturbance signal $v(t)$. It is furthermore assumed that $v(t)$ can be described as:

$$v(t) = H(q^{-1})\eta(t) \tag{10}$$

where $H(q^{-1})$ is a linear, asymptotically stable and inversely stable noise model, and $\eta(t)$ zero-mean white noise with variance σ^2.

As far as the notations are concerned, the signals collected under closed-loop operation using the controller $K(q^{-1}, \rho)$ will carry the argument ρ. The argument q^{-1} will be omitted when appropriate.

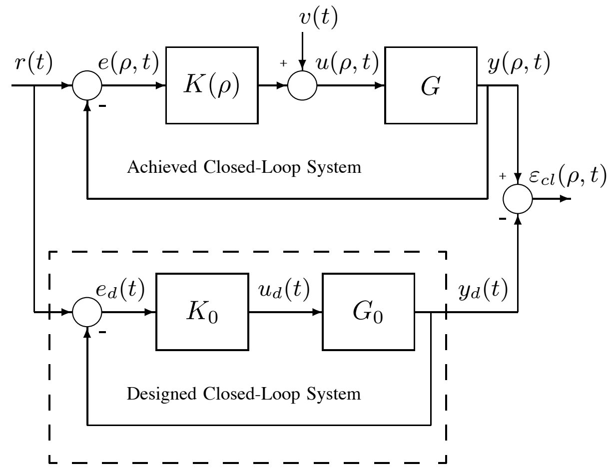

Fig. 1. Closed-loop output error resulting from a comparison of the achieved and designed closed-loop systems

Consider the model-following problem represented by the block diagram in Fig. 1. The upper part shows the achieved closed-loop system with the true plant G, while the lower part represents the designed closed-loop system that includes the plant model G_0 and the initial controller K_0. It is assumed that the initial controller is capable of meeting the specifications of the designed closed-loop system.

The closed-loop output error is defined as:

$$\varepsilon_{cl}(\rho, t) = y(\rho, t) - y_d(t) \tag{11}$$

where $y_d(t)$ is the output of the designed closed-loop system.

When applying the initial controller K_0 to the true plant excited by the reference signal $r(t)$, the closed-loop output error contains a contribution due to the difference between G and G_0 (modeling errors) and another contribution originating from the disturbance $v(t)$. The effect of modeling errors is correlated with the reference signal, whereas that of the disturbance is not. Therefore, adjusting the controller parameters to make the closed-loop output error $\varepsilon_{cl}(\rho, t)$ uncorrelated with the excitation signal $r(t)$ seems to be a reasonable tuning criterion. Ideally, the updated controller compensates the effect of modeling errors to the extent that the closed-loop output error contains only the filtered disturbance. For this purpose, let introduce the following n_ρ correlation equations:

$$f_N(\rho) = \frac{1}{N}\sum_{t=1}^{N} \zeta(\rho, t)\varepsilon_{cl}(\rho, t) = 0 \tag{12}$$

where N is the number of data points and $\zeta(\rho, t)$ a vector of instrumental variables that are correlated with the reference signal $r(t)$ and independent of the disturbance $v(t)$.

Since $f_N(\rho)$ depends in a complicated way on ρ, these equations have in general no explicit solution. However, they can be solved numerically using the Newton-Raphson iterative scheme:

$$\rho_{i+1} = \rho_i - \gamma_i \left[Q_N(\rho_i)\right]^{-1} f_N(\rho_i) \tag{13}$$

where γ_i is a scalar step size and $Q_N(\rho_i)$ is a square matrix of dimension n_ρ defined as follows:

$$Q_N(\rho_i) = \left.\frac{\partial f_N}{\partial \rho}\right|_{\rho=\rho_i} = \frac{1}{N}\sum_{t=1}^{N}\left\{ \left.\frac{\partial \zeta(\rho,t)}{\partial \rho}\right|_{\rho=\rho_i} \varepsilon_{cl}(\rho_i,t) + \zeta(\rho_i,t) \left.\frac{\partial \varepsilon_{cl}(\rho,t)}{\partial \rho}\right|_{\rho=\rho_i} \right\} \quad (14)$$

An accurate value of the Jacobian matrix $Q_N(\rho_i)$ cannot be computed because the derivative of $\varepsilon_{cl}(\rho,t)$ with respect to ρ is unknown. This derivative can be formally expressed as [1]:

$$\psi^T(\rho_i,t) = \left.\frac{\partial \varepsilon_{cl}(\rho,t)}{\partial \rho}\right|_{\rho=\rho_i} = \frac{B(q^{-1})}{P(q^{-1},\rho_i)}\phi^T(\rho_i,t) \quad (15)$$

where $P(q^{-1},\rho) = A(q^{-1})R(q^{-1},\rho) + B(q^{-1})S(q^{-1},\rho)$ is the closed-loop characteristic polynomial. Although the polynomials B and P are typically unknown, they can be identified and replaced by their estimates $\hat{B}$ and $\hat{P}$. Note that an estimate of the gradient can also be obtained by using one additional closed-loop experiment with one-degree-of-freedom controller operating in the loop [5].

In practice, precise knowledge of the Jacobian matrix is not important because a good estimate of this matrix is only required in the neighborhood of the solution [9]. The first term in (14) is close to zero because the derivatives of the instrumental variables are uncorrelated with the closed-loop output error near the solution. Neglecting this term and replacing $\psi(\rho_i,t)$ by its estimate $\hat{\psi}(\rho_i,t)$ leads to:

$$\hat{Q}_N(\rho_i) = \frac{1}{N}\sum_{t=1}^{N}\zeta(\rho_i,t)\hat{\psi}^T(\rho_i,t) \quad (16)$$

When the number of data points N tends to infinity, the iterative procedure (13) converges to the unique optimal solution ρ^o of the correlation equation (12) provided that it exists. However, for a finite N, a different solution of (12) results for each realization of $v(t)$, i.e. instead of a unique solution one has a set of solutions. This set is centered around ρ^o and its "size" depends strongly on the stochastic properties of the disturbance $v(t)$. The size of this set is characterized by the covariance of the correlation equation. In the next section, expressions that are asymptotic in N are derived for this covariance.

III. Asymptotic accuracy

Let the value of $f_N(\rho_i)$, when $N \to \infty$, be defined as:

$$f(\rho_i) = \lim_{N\to\infty} f_N(\rho_i) = E\left\{f_N(\rho_i)\right\} \quad (17)$$

There is no use of continuing the iterations if each element of the vector $f(\rho_i)$ is within a confidence interval defined by the corresponding element on the main diagonal of the covariance matrix of $f_N(\rho_i)$ at the optimal solution. Let assume that, based on this idea, after m iterations one has stopped iterating and the controller ρ_m is obtained. Then, from the expression for the covariance of $f_N(\rho^o)$, it is possible to calculate the asymptotic variance of the controller parameter estimates, as will be shown below. This variance, in turn, allows constructing the confidence ellipsoid around the optimal controller ρ^o that contains the controller ρ_m with the probability P_m. Now, using the explicit expression for the variance of the controller parameter estimates at the optimal solution, it is possible to construct around ρ_m a region containing the optimal controller ρ^o with the same probability P_m. This region could be interpreted as a controller uncertainty set. Expressions for this region that are asymptotic in N are derived in the sequel for both the controller parameters and the controller transfer function.

Let introduce the following assumptions:

- A1) The linear time-invariant SISO plant is strictly causal and of finite order.
- A2) The disturbance $v(t)$ defined in (10) is uncorrelated with the reference signal $r(\tau)$, $\forall t,\tau$.
- A3) The solution ρ^o of (12) exists and is unique (the corresponding controller will be called "optimal controller").
- A4) The controller computed at each iteration stabilizes the closed-loop plant.
- A5) The step size γ_i is constant and equal to 1 throughout the iterations.

Assumption A3 implies that the optimal controller belongs to the class of available controllers. Assumption A4 is somewhat restrictive but is necessary for being able to implement the controller calculated at each iteration.

A. Confidence interval

From expression (12) and the Central Limit Theorem [9], $\sqrt{N}f_N(\rho^o)$ tends in distribution to a normal distribution with zero mean and covariance P_f defined as:

$$P_f = \sigma^2 E\left\{\zeta_f(\rho^o,t)\zeta_f^T(\rho^o,t)\right\} \quad (18)$$

where $\zeta_f(\rho^o,t) = F(\rho^o)\zeta(\rho^o,t)$ with

$$F(\rho^o) = \frac{BR(\rho^o)}{P(\rho^o)}H = \sum_{i=0}^{\infty} f_i q^{-i} \quad (19)$$

Hence, one can test whether the k-th element of the correlation equation $f(k,\rho_i)$ falls inside the confidence interval one has:

$$|f(k,\rho_i)| \le \sqrt{\frac{\hat{P}_f(k,k)}{N}}\mathcal{N}_\alpha \quad \forall k = 1,\ldots,n_\rho \quad (20)$$

where

$$\hat{P}_f = \frac{\sigma^2}{N}\sum_{t=1}^{N}\zeta_f(\rho_i,t)\zeta_f^T(\rho_i,t) \quad (21)$$

and $\mathcal{N}_\alpha$ is the α-level of the normal distribution $\mathcal{N}(0,1)$. In practice, this test shows whether the selected controller order is appropriate. If $f(k,\rho_i)$, $\forall k = 1,\ldots,n_\rho$ does not enter the confidence interval after a large number of iterations, the controller order should be increased.

B. Asymptotic variance of controller parameter estimates

The variance of controller parameter estimates in the neighborhood of the optimal controller is calculated as follows. Assume that the optimal controller is used and one step of the iterative procedure is taken to produce the neighboring estimate ρ_{nb}:

$$\rho_{nb} = \rho^o - Q_N(\rho^o)^{-1} f_N(\rho^o) \tag{22}$$

The random variable $\rho_{nb} - \rho^o$ provides information regarding the accuracy of the method around the solution, and its asymptotic covariance matrix characterizes the region containing ρ_m. The following result can be obtained.

Theorem 3.1: Consider the iterative correlation-based controller tuning method (22). Suppose that the assumptions A1-A5 hold. Then, as the data length N tends to infinity, the distribution of the random variable $\sqrt{N}(\rho_{nb} - \rho^o)$ is asymptotically Gaussian:

$$\sqrt{N}(\rho_{nb} - \rho^o) \xrightarrow{dist} \mathcal{N}(0, P_{CbT}) \tag{23}$$

with the covariance matrix P_{CbT} given as follows:

$$P_{CbT} = Q(\rho^o)^{-1} P_f Q(\rho^o)^{-T} \tag{24}$$

where

$$Q(\rho^o) = \lim_{N\to\infty} Q_N(\rho^o) = E\left\{\zeta(\rho^o, t)\psi^T(\rho^o, t)\right\} \tag{25}$$

Proof: The proof goes along the ideas of Theorem 5.1 and its corollary in [13], p. 75, where the asymptotic distribution of parameter estimates for the extended IV open-loop estimator is investigated. Here, since the data are collected in closed loop, the transfer function between the white-noise input $\eta(t)$ and the output $y(t)$ is $F(\rho^o)$ defined in (19). Considering that the number of parameters is equal to the number of instrumental variables, the proof of the theorem follows easily. ■

In practice, (24) can be evaluated by replacing the optimal controller parameter vector ρ^o by the current value ρ_i. In the same way, since the exact value of $Q(\rho^o)$ is unknown its estimate $\hat{Q}_N(\rho_i)$ is calculated using expression (16).

Equations (18), (19) and (24) show that the covariance matrix P_{CbT} depends on the choice of the instrumental variable ζ in a rather complex way. In addition, P_{CbT} depends on the noise model $H(q^{-1})$ and the true plant. Since both are unknown, this makes accuracy optimization quite involved. Fortunately, a solution to this problem has already been proposed in the field of system identification [13], and it will be detailed in the sequel.

C. Optimal choice of instrumental variables

In this section, a lower bound for P_{CbT} is established, and then the choice of instrumental variables that makes this bound achievable is presented.

Let denote by $\tilde{\phi}(\rho^o, t)$ the noise-free part of the regressor vector $\phi(\rho^o, t)$:

$$\tilde{\phi}(\rho^o, t) = [\tilde{\varphi}^T(\rho^o, t-1), \cdots, \tilde{\varphi}^T(\rho^o, t-n)]^T \tag{26}$$

with

$$\tilde{\varphi}(\rho^o, t-1) = \left(-\frac{AS(\rho^o)}{P(\rho^o)} r(t-1), \frac{AR(\rho^o)}{P(\rho^o)} r(t)\right)^T \tag{27}$$

Theorem 3.2: P_{CbT} given in (24) is bounded from below by:

$$P_{lo} = \sigma^2 E\left\{\frac{1}{R(\rho^o)H}\tilde{\phi}(\rho^o, t)\frac{1}{R(\rho^o)H}\tilde{\phi}^T(\rho^o, t)\right\}^{-1} \tag{28}$$

Moreover, $P_{CbT} = P_{lo}$ when the following relationship holds:

$$\sum_{i=0}^{\infty} \zeta(\rho^o, t+i) f_i = \frac{1}{R(\rho^o)H}\tilde{\phi}(\rho^o, t) \tag{29}$$

Proof: see Appendix I. ■

From (29), it is obvious that the choice of instruments

$$\zeta_{opt}(\rho^o, t) = \frac{1}{R(\rho^o)HF(\rho^o)}\tilde{\phi}(\rho^o, t) \tag{30}$$

provides optimal accuracy. However, in order to implement ζ_{opt} throughout the iterations, one has to estimate the models of the noise H and the plant G, which imposes additional computational effort to the algorithm. In addition, in the filter in expression (30), the optimal parameters ρ^o need to be replaced by the current values ρ_i. This seems to be a reasonable approximation considering the assumption that the current controller is in the neighborhood of the optimal controller.

Note that the data collected in closed loop can be filtered by some linear filter. This way, additional design variables are available to improve accuracy. However, this issue will not be addressed in this paper.

D. Asymptotic variance of transfer function estimate

This section derives the variance expression for the transfer function estimate. This expression is asymptotic in both the number of data points and the model order.

Since ρ_{nb} is in vicinity of ρ^o, it follows from (24) and Gauss approximation formula [9] that

$$\sqrt{N}(K(\rho_{nb}) - K(\rho^o)) \xrightarrow{dist} \mathcal{N}(0, \mathcal{P}_n(\omega)) \tag{31}$$

with

$$\mathcal{P}_n(\omega) = T(\omega, \rho^o) P_{CbT} T^T(-\omega, \rho^o) \tag{32}$$

where $T(\omega, \rho^o)$ is a $2n$-dimensional row vector representing the derivatives of $K(\rho^o)$ with respect to ρ.

The expression (32) is asymptotic in N, but exact in n. A simpler expression can be obtained for $n \to \infty$. The result is given in the following proposition.

Proposition 3.1: As n and N tend to infinity, the variance of the controller transfer function becomes:

$$\text{Var}\, K(\rho^o) \approx \frac{n}{N}\frac{\Phi_v(\omega)}{\Phi_e(\omega)} \tag{33}$$

Proof: see Appendix II. ■

The result (33) is interesting and not at all surprising. In fact, the variance is proportional to the ratio of the noise spectrum to the control error spectrum (Φ_v/Φ_e), with the factor of proportionality being the ratio of the number of parameters to the number of data points (n/N). This result is exactly dual to that in system identification [8], [9], where the covariance of the plant model is proportional to the spectral ratio of the plant output noise and the plant input signal, with the same factor of proportionality. Note also that the estimate $K(\rho^o)$ is asymptotically based on only input-output properties at the frequency ω, i.e. independent of the choice of the instrumental variables.

IV. Conclusions

The accuracy aspects of the iterative correlation-based controller tuning approach have been studied. The confidence interval for the correlation equation has been derived and an asymptotic expression for the covariance around the optimal controller has been given. This expression is used to construct the optimal instruments. In practice, one will stop iterating when the correlation equation remains within the confidence interval for several consecutive iterations. Then, the region around the resulting controller contains the optimal controller with a certain probability level. This region can be reduced by increasing the number of data points. In addition, expression (33) shows that this region can be reduced around a particular frequency by applying appropriate excitation signal.

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Appendix I
Proof of Theorem 3.2

This is similar to the proof of Theorem 6.1 in [13], p. 96. Taking into account that $\zeta(\rho^o,t)$ is uncorrelated with $v(t)$, it follows from (25) with (15) and (19):

$$\begin{aligned}
Q(\rho^o) &= E\left\{\zeta(\rho^o,t)\frac{B}{P(\rho^o)}\tilde{\phi}^T(\rho^o,t)\right\} \\
&= E\left\{\zeta(\rho^o,t)\sum_{i=0}^{\infty} f_i q^{-i}\frac{B}{P(\rho^o)F(\rho^o)}\tilde{\phi}^T(\rho^o,t)\right\} \\
&= E\left\{\sum_{i=0}^{\infty}\zeta(\rho^o,t)f_i\frac{1}{R(\rho^o)H}\tilde{\phi}^T(\rho^o,t-i)\right\} \\
&= E\left\{\left[\sum_{i=0}^{\infty}\zeta(\rho^o,t+i)f_i\right]\frac{1}{R(\rho^o)H}\tilde{\phi}^T(\rho^o,t)\right\} \qquad (34)
\end{aligned}$$

Furthermore, the assumption of stationarity gives:

$$\frac{P_f}{\sigma^2} = E\left\{\sum_{i=0}^{\infty}\zeta(\rho^o,t+i)f_i \times \sum_{l=0}^{\infty}\zeta(\rho^o,t+l)^T f_l\right\} \qquad (35)$$

The matrix inequality

$$E\left\{\begin{bmatrix}\frac{1}{R(\rho^o)H}\tilde{\phi}(\rho^o,t)\\ \sum_{i=0}^{\infty}\zeta(\rho^o,t+i)f_i\end{bmatrix} \times \begin{bmatrix}\frac{1}{R(\rho^o)H}\tilde{\phi}^T(\rho^o,t) & \sum_{i=0}^{\infty}\zeta^T(\rho^o,t+i)f_i\end{bmatrix}\right\} \geq 0 \qquad (36)$$

can equivalently be expressed as:

$$\begin{aligned}
&E\left\{\frac{1}{R(\rho^o)H}\tilde{\phi}(\rho^o,t)\frac{1}{R(\rho^o)H}\tilde{\phi}^T(\rho^o,t)\right\} \\
&\quad - E\left\{\frac{1}{R(\rho^o)H}\tilde{\phi}(\rho^o,t)\sum_{i=0}^{\infty}\zeta^T(\rho^o,t+i)f_i\right\} \\
&\quad \times E\left\{\sum_{i=0}^{\infty}\zeta(\rho^o,t+i)f_i\sum_{i=0}^{\infty}\zeta^T(\rho^o,t+i)f_i\right\}^{-1} \\
&\quad \times E\left\{\sum_{i=0}^{\infty}\zeta(\rho^o,t+i)f_i\frac{1}{R(\rho^o)H}\tilde{\phi}^T(\rho^o,t)\right\} \geq 0 \qquad (37)
\end{aligned}$$

Now, from (24), (28), (34), (35), and (37), it follows that $P_{CbT}^{-1} \le P_{lo}^{-1}$. Finally, it is easy to verify that $P_{CbT} = P_{lo}$ when (29) holds. ∎

APPENDIX II
PROOF OF PROPOSITION 3.1

The following lemma from [15] will be used.

Lemma 2.1: Let $R_n^{(l)}$ be a $2n \times 2n$ block-Toeplitz matrix where the $(i-j)$ 2×2 block is $r_l(i-j)$. Let

$$\Phi_l(\omega) = \sum_{\tau=-\infty}^{\infty} r_l(\tau) e^{-j\omega\tau} \quad l = 1, 2$$

Then

$$\lim_{n\to\infty} \frac{1}{n} W_n(\omega)[R_n^{(1)}]^{-1} R_n^{(2)} W_n^T(-\omega) = [\Phi_1(\omega)]^{-1}\Phi_2(\omega)$$

Proof: see the proof of Lemma 4.3 in [15]. ∎

Let introduce the vector of instrumental variables

$$\zeta^T(t) = \left[z^T(t-1), z^T(t-2), \cdots, z^T(t-n)\right] \quad (38)$$

where

$$z^T(t) = \left[Z_1(q^{-1})r(t), Z_2(q^{-1})r(t)\right] = Z(q^{-1})r(t). \quad (39)$$

The derivative of $K(\rho^o)$ w.r.t. ρ can be expressed as:

$$T(\omega, \rho^o) = D(\omega, \rho^o) W_n(\omega) \quad (40)$$

where

$$\begin{aligned} D(q^{-1}, \rho^o) &= \left[-\frac{S(q^{-1}, \rho^o)}{R^2(q^{-1}, \rho^o)}, \frac{q}{R(q^{-1}, \rho^o)} \right] \\ &= \frac{1}{R(q^{-1}, \rho^o)} \Gamma^T(\rho^o) \end{aligned} \quad (41)$$

with $\Gamma^T(\rho^o) = \left[-K(q^{-1}, \rho^o) \quad q\right]$, and

$$W_n(\omega) = \left[e^{-j\omega} I \quad e^{-2j\omega} I \quad \cdots \quad e^{-nj\omega} I\right] \quad (42)$$

I being the 2×2 identity matrix. Assume that a regularizing term λI is added to the right-hand side of (14) and the resulting $Q(\rho^o)$ is used to calculate P_{CbT} in (24).

The elements $\varphi(\rho^o, t)$ of the regression vector $\phi(\rho^o, t)$ can be expressed as:

$$\varphi(\rho^o, t) = \frac{AR(\rho^o)}{P(\rho^o)} \Gamma(\rho^o) r(t) = S_{yp} \Gamma(\rho^o) r(t) \quad (43)$$

with S_{yp} being the output sensitivity function. For the sake of simplicity of notations, the argument ρ^o is omitted in the sequel whenever appropriate.

The cross-spectrum between $\zeta(t)$ and $\psi(t)$ reads:

$$\Phi_{\zeta\psi}(\omega) = Z(e^{-i\omega}) S_{yp}(e^{i\omega}) \frac{B(e^{i\omega})}{P(e^{i\omega})} \Gamma^T(e^{i\omega}) \Phi_r(\omega) \quad (44)$$

Similarly, the Fourier transform of the Toeplitz matrix P_f in (18) reads:

$$\Phi_{\zeta_f}(\omega) = \sigma^2 \left|F(e^{-i\omega})\right|^2 Z(e^{-i\omega}) Z^T(e^{i\omega}) \Phi_r(\omega) \quad (45)$$

Finally, applying lemma 2.1 twice to the inner product of (32) and (40) gives:

$$\begin{aligned} \mathcal{M}_\lambda(\omega) &= \lim_{n\to\infty} \frac{1}{n} W_n(\omega) P_{CbT} W_n^T(-\omega) \\ &= \left(Z S_{yp}^* \frac{B^*}{P^*} \Gamma^{*T} \Phi_r + \lambda I \right)^{-1} \times \Phi_{\zeta_f} \\ &\times \left(S_{yp} \frac{B}{P} \Gamma Z^{*T} \Phi_r + \lambda I \right)^{-1} \end{aligned} \quad (46)$$

where the arguments are omitted for the sake of simplicity and asterisk is used to denote the complex conjugate. After straightforward but tedious calculations, the expression from (46) can be reformulated as

$$\mathcal{M}_\lambda(\omega) = \frac{\sigma^2 |F|^2}{|S_{yp}|^2 |\frac{B}{P}|^2 \Phi_r} \frac{Z Z^{*T}}{|\Gamma Z^{*T} + \lambda I/(S_{yp}\frac{B}{P}\Phi_r)|^2} \quad (47)$$

Now, from (19), (32), (40), (41) and (47), one has

$$\begin{aligned} \mathcal{P}(\omega) &= \lim_{\lambda\to 0} \lim_{n\to\infty} \frac{1}{n} \mathcal{P}_n(\omega, \lambda) \\ &= \lim_{\lambda\to 0} D(w, \rho^o) \mathcal{M}_\lambda(\omega) D^T(-w, \rho^o) \\ &= \lim_{\lambda\to 0} \frac{\Gamma^T}{R} \frac{\sigma^2 |RH|^2}{|S_{yp}|^2 \Phi_r} \\ &\times \frac{Z Z^{*T}}{|\Gamma Z^{*T} + \lambda I/(S_{yp}\frac{B}{P}\Phi_r)|^2} \frac{\Gamma^*}{R^*} \\ &= \frac{\sigma^2 |H|^2}{|S_{yp}|^2 \Phi_r} \end{aligned} \quad (48)$$

Combining this expression with (10), one finally obtains:

$$\mathcal{P}(\omega) = \frac{\Phi_v(\omega)}{\Phi_e(\omega)} \quad (49)$$

where Φ_v denotes the spectrum of the random process $v(t)$ and Φ_e the spectrum of the control error:

$$\Phi_e(\omega) = |S_{yp}(\omega)|^2 \Phi_r(\omega)$$

The expression for the asymptotic variance of $K(\rho^o)$ follows readily from (49). ∎