An insensitivity property of Lundberg's estimate for delayed claims

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Abstract: This short note shows that the Lundberg's exponential upperbound in the ruin problem of non-life insurance with compound Poisson claims is also valid for the Poisson shot noise delayed claims model, and that the optimal exponent depends only on the distribution of the total claim per accident, not on the time it takes to honor the claim. This result holds under Cramer's condition (For background in the classical Poisson compound model of claims, see the first pages of [2] or[3], and for the delayed claims problem, see [4] where the shot noise model is studied.)

1 Poisson Shot Noise

A Poisson shot noise (PSN) is a process of the form

$$X(t) = \sum_n h(t - T_n, X_n)\ 1_{(0,t]}(T_n),$$

where $\{T_n\}_{n\in\mathbb{Z}}$ is the sequence of times of a homogeneous Poisson process of rate λ, and $\{X_n\}_{n\in\mathbb{Z}}$ is an i.i.d sequence of random elements of a measurable space $(E, \mathcal{E})$, independent of the Poisson process, and $h(t, x) = 0$ for negative times, and is otherwise non-negative. In

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the sequel, the time t is always positive. Campbell's formula

$$E[X(t)] = \lambda \int_0^\infty E[h(s, X_1)]ds,$$

an therefore, the PSN, which is always well defined since the function h is non-negative, is finite if

$$\rho \stackrel{\text{def}}{=} \lambda \int_0^\infty E[h(t, X_1)]dt < \infty.$$

A PSN is also called a (randomly) filtered Poisson process, with the interpretation that $h(t - T_n, X_n)$ is the random impulse response associated with spike, or impulse, at T_n.

The total (or integrated) input in the time interval $(0, t]$ is

$$A((0,t]) \; = \; \int_0^t X(s)ds,$$

and it takes the form

$$A((0,t]) \; = \; \sum_n H(t - T_n, X_n)\, 1_{(0,t]}(T_n) + \sum_n (H(t - T_n, X_n) - H(-T_n, X_n))\, 1_{(-\infty,0]}(T_n), \tag{1.1}$$

where

$$H(t,x) \; = \; \int_0^t h(s,x)ds.$$

The interpretation of this model in terms of non-life insurance is the following. A claim occurs at time T_n and the insurance company honors this claim at the rate $h(t - T_n, X_n)$. The total claim is therefore $\sigma_n = \int_0^\infty h(t - T_n, X_n)dt$. This is a fluid model of claims.

To account for claims which not necessarily fluid (as the compound Poisson model of the classical theory), we adopt the integrated Poisson shot noise (IPSN) model (1.1) where

$$H(t,x) \; = \; \mu(x, [0,t]) \tag{1.2}$$

and $\mu(x, \cdot)$ is for each x a measure on the non-negative half-line.

If $\mu(x, \cdot) = x\delta(\cdot)$ where $\delta(\cdot)$ is the Dirac unit mass at 0, we have the input of the classical compound Poisson model. In this case we denote X_n by σ_n, and therefore

$$A((0,t]) \; = \; \sum_n \sigma_n 1_{(0,t]}(T_n).$$

In the general model

$$E[A([0,t))] \; = \; \lambda \int_0^t E[H(s, X_1)]ds \; + \; \lambda \int_0^\infty E[H(t+s, X_1) - H(s, X_1)]ds.$$

In particular if this quantity is finite for some $t > 0$, then it is finite for all $t > 0$, and proportional to t. The traffic intensity is then

$$\rho \stackrel{\text{def}}{=} \frac{1}{t}\lambda \left\{ \int_0^t E[H(s,X_1)]ds + \int_0^\infty E[H(t+s,X_1) - H(s,X_1)]ds \right\}. \tag{1.3}$$

The total claim for accident 1 is

$$\sigma_1 = H(\infty, X_1) \;=\; \mu(X_1, \mathbb{R}_+).$$

2 Lundberg's estimate

The insurance company starts with an initial fortune u. The total amount paid by the insurance company in the interval $(0,t]$ is

$$A_0((0,t]) \;=\; \sum_n H(t - T_n, X_n)\, 1_{(0,t]}(T_n). \tag{2.1}$$

The ruin probability is therefore, when the gross premium rate is c,

$$\Psi^{SN}(u) \;=\; P(\sup_{t\geq 0}\{A_0((0,t]) - ct \geq u\},$$

where the upper index refers to shot noise. In the classical ruin problem, with compound Poisson claim process, we omit the upper index. We assume the usual condition

$$\rho \;<\; c. \tag{2.2}$$

We also assume that the claim process is light-tailed, that is

$$E\left[e^{\theta H(\infty, X_1)}\right] \;<\; \infty, \tag{2.3}$$

for all θ in a neighborhood of 0. This is Cramer's condition.

We have the following result:

Theorem 2.1. Under Cramer's condition, that is if there exists $R > 0$ solution of

$$\lambda E[\sigma_1] \int_0^\infty e^{Rz} dF_I(z) = c, \tag{2.4}$$

where $F_I(t) = (E[\sigma_1])^{-1} \int_0^t (1 - F(z))dz$ is the stationary residual claim distribution, then we have Lundberg's inequality

$$\Psi^{SN}(u) \;\leq\; e^{-Ru}. \tag{2.5}$$

Moreover R is the best exponent in Lundberg's inequality for the delayed claims process.

Proof. If in the classical (non-delayed) model we take for claim size $\sigma_1 = H(\infty, X_1)$, we have the inequality

$$\Psi^{SN}(u) \ \leq \ \Psi(u), \tag{2.6}$$

which results from the observation that the classical ruin process is dominated by the corresponding delayed claims ruin process.

To prove that R is the best exponent in Lundberg's inequality for the delayed claims process we begin by showing that

$$\lim_{u\uparrow\infty} \frac{1}{u}\Psi^{SN}(u) = -R',$$

where R' is the solution of

$$\lambda E\left[e^{R'\sigma_1} - 1\right] \ = \ cR'. \tag{2.7}$$

To show this, we use the general results of [1], which give

$$R' = \sup\{\theta\ ;\ \lambda(\theta) \leq 0\}, \tag{2.8}$$

where

$$\lambda(\theta) \ = \ \lim_{t\uparrow\infty} \frac{1}{t} \ln E\left[e^{\theta(A_0([0,t))-ct)}\right]. \tag{2.9}$$

Straightforward computations (see section 2.1 of [4]) show that

$$\frac{1}{t} \ln E[e^{\theta A_0([0,t))}] = \lambda \frac{1}{t} \int_0^t E[e^{\theta H(s,X_1)} - 1]ds\ . \tag{2.10}$$

Under the light-tail assumption (2.3), the first term in the left-hand side converges to

$$\lambda E[e^{\theta H(\infty,X_1)} - 1].$$

It now remains to show that $R = R'$. But this follows from the observation that

$$\lambda \int_0^\infty e^{\theta z}(1 - F(z))dz == \frac{\lambda}{\theta} E\left[e^{\theta\sigma_1} - 1\right].$$

□

References

[1] Duffield, N.G. and N. O'Connell (1995) Large deviations and overflow probabilities for the general single-server queue, with applications, Math. Proc. Camb. Phil. Soc., 118, 363-374

[2] Embrechts, P., Kluppelberg, C. and T. Mikosch (1997) *Modelling Extremal Events*, Springer, NY

[3] Grandell, J. (1992) *Aspects of Risk Theory*, Springer-Verlag, NY

[4] Kluppelberg, C. and T. Mikosch (1995) Explosive Poisson shot noise processes with applications to risk reserves, Bernoulli, 1, 1/2, 125-147