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Groebner basis methods for multichannel sampling with unknown offsets ☆

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Abstract

In multichannel sampling, several sets of sub-Nyquist sampled signal values are acquired. The offsets between the sets are unknown, and have to be resolved, just like the parameters of the signal itself. This problem is nonlinear in the offsets, but linear in the signal parameters. We show that when the basis functions for the signal space are related to polynomials, we can express the joint offset and signal parameter estimation as a set of polynomial equations. This is the case for example with polynomial signals or Fourier series. The unknown offsets and signal parameters can be computed exactly from such a set of polynomials using Gröbner bases and Buchberger's algorithm. This solution method is developed in detail after a short and tutorial overview of Gröbner basis methods. We then address the case of noisy samples, and consider the computational complexity, exploring simplifications due to the special structure of the problem.

Keywords: Aliasing; Sampling; Multichannel sampling; Offset estimation; Groebner basis

1. Introduction

If a bandlimited signal is uniformly sampled at a frequency lower than twice its maximum frequency, the sampled signal is aliased, and perfect reconstruction is generally not possible. This is a well-known result from the Shannon–Nyquist sampling theorem [19]. However, if multiple uniform sets of samples with small relative offsets are available, the original signal can be reconstructed from the combined set of all samples. Such a setup is often called multichannel sampling, and was first studied by Papoulis [16]. The results were later extended by Unser and Zerubia [23,24] in their generalized sampling theory.

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These methods can be applied to high-rate A/D converters, which use multiple parallel A/D converters at a lower rate, operating with small relative offsets [6,10]. Similarly, on two-dimensional signals, these techniques can be applied in super-resolution imaging. Super-resolution techniques use multiple images taken from almost the same point of view to reconstruct a higher resolution image [8,18,28].

However, in most of these applications, the relative offsets between the different sets of samples are unknown. In this paper, we will therefore study reconstruction methods for multiple aliased sets of samples with unknown offsets. That is, we have to solve for *both the unknown signal coefficients and the unknown offsets*. A method to solve this problem for discrete-valued offsets is presented by Marziliano and Vetterli [14]. For the reconstruction problem with unknown, continuous-valued offsets, Vandewalle et al. [26] give a solution using projections onto subspaces.

A first contribution of this paper is to show that, in many cases, the multichannel sampling problem with unknown offsets can be written as a set of polynomial equations in both the unknown signal coefficients and the offsets. The solution can then be computed using Gröbner bases. In any practical setting, the samples are corrupted by noise, and then there is no algebraic solution. Thus, a second contribution of the paper is to address this noisy version of the problem, and to show how a good approximation can be obtained from multiple Gröbner bases for subsets of samples.

Gröbner basis theory is a very powerful tool from algebraic geometry. The theory was originally introduced by Buchberger in 1965 [1], and can be found in some very good text books, like for example the book by Cox et al. [5], as well as in many free (Macaulay2, Singular) and commercial (Mathematica, Maple, Magma) software packages. Gröbner bases have also found their way into many applications in signal processing and system theory [2,3]. Examples can be found in filter bank design [4,9,12,17], multichannel deconvolution [29], or motion estimation [11]. In this last paper, Holt et al. use algebraic geometry to determine the number of solutions and uniqueness for certain problems in three-dimensional motion estimation. They analyze the 3D motion of a rigid link moving in a plane where one endpoint is known, and the extraction of 3D motion from 2D optical flow information. In this paper, we will consider shifts of one-dimensional signals, which can be extended to global planar shifts of images in the image plane.

This paper is structured as follows. The multichannel sampling problem with unknown offsets is formulated mathematically as a set of polynomial equations in Section 2. Section 3 gives an overview of Gröbner basis theory, and more particularly the main ideas that we will use for our reconstruction problem. Gröbner bases are then applied to the multichannel sampling problem in Section 4. Section 5 presents a solution for noisy measurements. The complexity of such an algorithm is discussed in Section 6, and some optimizations are presented that take advantage of the particular structure of the polynomials. Finally, Section 7 concludes the paper.

2. Problem setup

A mathematical formulation of the multichannel sampling problem presented in the introduction is given below. This setup is the same as the one used in [25], so the reader can find a more detailed description and some more examples in that reference.

Let us consider a finite L-dimensional Hilbert space $\mathcal{H}$ with basis $\{\varphi_l(t)\}_{l=0,\dots,L-1}$ ($\mathcal{H} = \mathrm{span}(\{\varphi_l(t)\}_{l=0,\dots,L-1})$). For simplicity, assume the space to be periodic, of period 1. The time t can then be taken modulo 1, and we restrict our analysis to the interval $[0, 1)$. An arbitrary signal $f(t)$ in $\mathcal{H}$ can then be written as

$$f(t) = \sum_{l=0}^{L-1} \alpha_l \varphi_l(t), \tag{1}$$

with α_l the expansion coefficient corresponding to the basis function $\varphi_l(t)$. We sample $f(t)$ uniformly with N samples, resulting in

$$y_0(n) = f\left(\frac{n}{N}\right) = \sum_{l=0}^{L-1} \alpha_l \varphi_l\left(\frac{n}{N}\right) \quad \text{for } 0 \leqslant n < N. \tag{2}$$

If we choose the number of samples $N < L$, it is not possible to compute the L expansion coefficients α_l from the N samples $y_0(n)$. We will therefore consider M such sets of samples, with for each set a relative offset t_m ($0 \leqslant m < M$ and $t_0 = 0$). For every additional set of samples, we obtain in this way N new equations, while adding only a single unknown t_m. A sample from the mth set can be written as

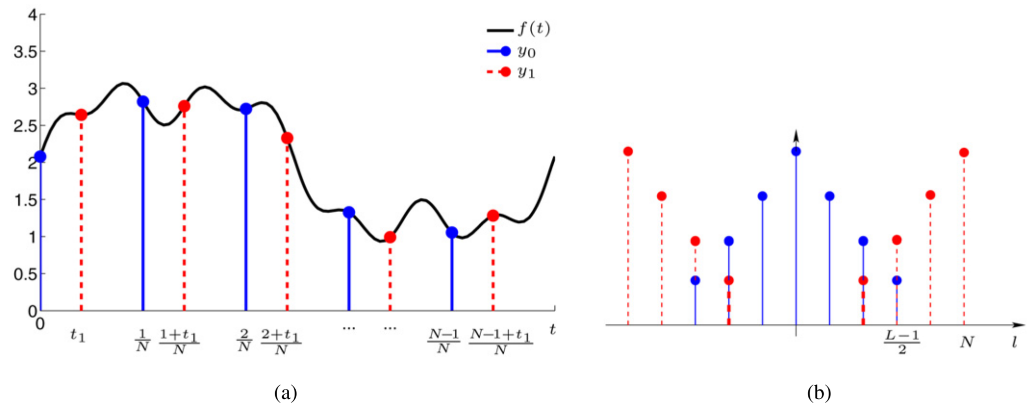

Fig. 1. Illustration of the different variables with $M = 2$ sets of samples and a Fourier basis. (a) The original signal $f(t)$ has L Fourier coefficients (L odd), extending from $-(L-1)/2$ to $(L-1)/2$. $\mathcal{H}$ is sampled at times n/N for the first set y_0 (—), and at $(n+t_1)/N$ for the second set y_1 (– –). (b) Frequency domain representation of the absolute values of the signal spectrum (—) and its aliased copies after sampling (– –).

$$y_m(n) = f\left(\frac{n+t_m}{N}\right) = \sum_{l=0}^{L-1} \alpha_l \varphi_l\left(\frac{n+t_m}{N}\right). \tag{3}$$

This setup is illustrated in Fig. 1. If we combine all the samples from the mth set in a vector $\mathbf{y}_m$, this can be rewritten as

$$\mathbf{y}_m = \boldsymbol{\Phi}_{t_m} \boldsymbol{\alpha}, \tag{4}$$

with $\boldsymbol{\alpha}$ the vector containing the expansion coefficients α_l, and $\boldsymbol{\Phi}_{t_m}$ an $N \times L$ matrix with the sampled basis functions as its columns. Putting all the sets of samples $\mathbf{y}_m$ together into a single vector $\mathbf{y}$ of length MN, and similarly combining all $\boldsymbol{\Phi}_{t_m}$ into the $MN \times L$ matrix $\boldsymbol{\Phi}_{\mathbf{t}}$, we obtain:

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_{M-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Phi}_{t_0} \\ \boldsymbol{\Phi}_{t_1} \\ \vdots \\ \boldsymbol{\Phi}_{t_{M-1}} \end{pmatrix} \boldsymbol{\alpha} = \boldsymbol{\Phi}_{\mathbf{t}} \boldsymbol{\alpha}. \tag{5}$$

Example 2.1 *(Second degree polynomials).* Let us illustrate this setup with an example. Consider the space $\mathcal{H}$ defined as the span of the functions $\varphi_0(t) = t^2$, $\varphi_1(t) = t$, $\varphi_2(t) = 1$. Assume that we take two sets of two samples, i.e. $M = 2$, $N = 2$. If we consider the signal parameters $\boldsymbol{\alpha} = (64, -24, -4)^T$ and offsets $\mathbf{t} = (0, 1/4)$, the two sets of samples are $\mathbf{y}_0 = (-4, 0)^T$ and $\mathbf{y}_1 = (-6, 6)^T$. The signal and its samples are shown in Fig. 2. In this case, (5) becomes

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{2} & 1 \\ \frac{1}{4}t_1^2 & \frac{1}{2}t_1 & 1 \\ (\frac{1}{2} + \frac{1}{2}t_1)^2 & \frac{1}{2} + \frac{1}{2}t_1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -6 \\ 6 \end{pmatrix}. \tag{6}$$

We can clearly see that the unknown offset t_1 appears together in the equations with the unknown signal coefficients α_0, α_1 and α_2.

In the above example, we obtain a set of nonlinear polynomial equations. The equations are linear in the unknown signal coefficients $\boldsymbol{\alpha}$. Thanks to the specific choice of a polynomial basis, the equations are polynomials in the offsets $\mathbf{t}$. Note that for an arbitrary basis $\{\varphi_l(t)\}$, this is not valid. However, for certain bases, we can rewrite (5) as a set of polynomial equations using a change of variables. This is possible when the basis is a set of functions $\varphi_l(t) = h(t)^l$, with $h(t)$ an invertible function.

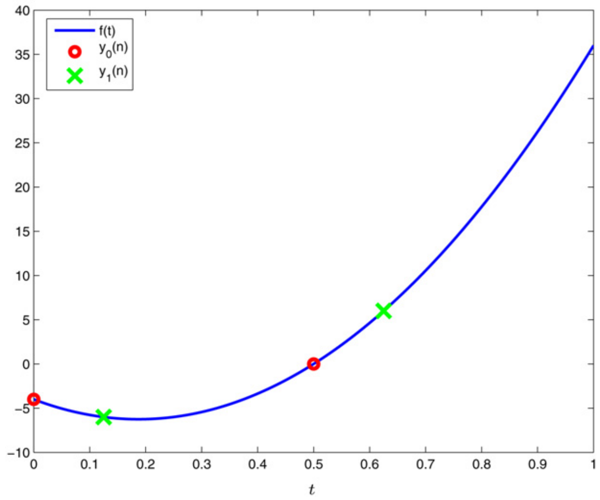

Fig. 2. Second degree polynomial signal used in Example 2.1. The signal $f(t) = 64t^2 - 24t - 4$ is sampled with two sets of two samples $\mathbf{y}_0 = (-4, 0)^T$ and $\mathbf{y}_1 = (-6, 6)^T$ with offset $t_1 = 1/4$.

Probably the most important and practically useful example of such a basis is when $h(t) = e^{j2\pi t}$, that is, the Fourier series. In fact, consider the case of a complex signal of the form

$$f(t) = \sum_{l=-K}^{K} \alpha_l \varphi_l(t), \tag{7}$$

with $\varphi_l(t) = e^{j2\pi lt}$. Note that the basis functions and coefficients are now indexed from $-K$ to K (instead of 0 to $L-1$ previously), which is the usual way of indexing for Fourier series. For coherence with the previous example, we will assume here that $K = (L-1)/2$, with L odd. The samples are given by

$$y_m(n) = f\left(\frac{n+t_m}{N}\right) = \sum_{l=-K}^{K} \alpha_l W^{nl} e^{\frac{j2\pi l t_m}{N}} \quad \text{for } 0 \leqslant n < N, \tag{8}$$

with $W = e^{j2\pi/N}$. By setting $z_m = e^{j2\pi t_m/N}$, we obtain

$$y_m(n) = f\left(\frac{n+t_m}{N}\right) = \sum_{l=-K}^{K} \alpha_l W^{nl} z_m^l. \tag{9}$$

We multiply (9) with z_m^K to eliminate negative exponents:

$$z_m^K y_m(n) = z_m^K f\left(\frac{n+t_m}{N}\right) = \sum_{l=-K}^{K} \alpha_l W^{nl} z_m^{l+K}. \tag{10}$$

For each sample, this can be rewritten as a polynomial constraint

$$p_{nN+m} = \sum_{l=-K}^{K} \alpha_l W^{nl} z_m^{l+K} - z_m^K y_m(n) = 0. \tag{11}$$

In this equation, the unknowns are the signal parameters α_l and the offset-dependent variables z_m. As in Example 2.1, the equations are linear in the signal parameters and polynomial in the offset variables z_m. We will now introduce Gröbner bases and Buchberger's algorithm, which provide an elegant method to solve such a set of polynomial equations.

3. Gröbner bases

It is beyond the scope of this paper to give a complete presentation of algebraic geometry and Gröbner bases. We present here the main results related to our multichannel sampling problem and we refer to Cox et al. [5] and Buchberger [2,3] for a complete presentation of algebraic geometry and Gröbner bases. This section is intended as a quick introduction and overview of key results that are necessary to our solution method. It can be skipped by readers familiar with Gröbner bases.

3.1. Affine varieties and ideals

We consider polynomials in the n complex variables, $x_0, \dots, x_{n-1}$. A polynomial p can then be written compactly as

$$p = \sum_{\mathbf{d}} a_{\mathbf{d}} \mathbf{x}^{\mathbf{d}}, \quad a_{\mathbf{d}} \in \mathbb{C}, \tag{12}$$

where the sum is over a finite number of n-tuples $\mathbf{d} = (d_0, \dots, d_{n-1})$ and $\mathbf{x}^{\mathbf{d}}$ is a compact notation for $x_0^{d_0} \dots x_{n-1}^{d_{n-1}}$. Each term of the sum in (12) is called a *monomial*. In the following, we will denote $\mathbb{C}[x_0, \dots, x_{n-1}]$ the set of (complex) polynomials in the variables $x_0, \dots, x_{n-1}$.

The basic objects of algebraic geometry are affine varieties:

Definition 3.1 *(Affine variety).* Consider the polynomials $p_0, \dots, p_{s-1}$ in the n variables $x_0, \dots, x_{n-1} \in \mathbb{C}$. Then we set

$$V(p_0, \dots, p_{s-1}) = \{(c_0, \dots, c_{n-1}) \in \mathbb{C}^n\colon\ p_i(c_0, \dots, c_{n-1}) = 0,\ \forall 0 \leqslant i < s\}. \tag{13}$$

We call $V(p_0, \dots, p_{s-1})$ the *affine variety* defined by $p_0, \dots, p_{s-1}$. The elements of an affine variety are the points for which the polynomials $p_0, \dots, p_{s-1}$ are all zero.

The determination of the affine variety is trivial in the linear case, since the polynomial p_i has the simple form

$$p_i(x_0, \dots, x_{n-1}) = a_{i0}x_0 + \dots + a_{i(n-1)}x_{n-1} + b_i, \quad i = 0, \dots, s-1, \tag{14}$$

and the points of the variety $V(p_0, \dots, p_{s-1})$ are those that satisfy the system

$$\mathbf{A}\mathbf{x} + \mathbf{b} = 0, \tag{15}$$

with $\{\mathbf{A}\}_{i,j} = a_{ij}$ and $\mathbf{b} = (b_0, \dots, b_{s-1})^T$. The solution can be easily computed by using Gaussian elimination. Recall that Gaussian elimination consists in computing linear combinations of the rows of (15) in order to remove progressively the variables. The method is based on a certain ordering of the variables. For example, with the ordering $x_0, x_1, \dots, x_{n-1}$, we obtain a system

$$\tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{b}} = \mathbf{0}. \tag{16}$$

The ith row of $\tilde{\mathbf{A}}$ has the form

$$(\,0 \quad \dots \quad 0 \quad \tilde{a}_{ij_i} \quad \tilde{a}_{ij_i+1} \quad \dots \quad \tilde{a}_{i(n-1)}\,). \tag{17}$$

The leading zeros in each row correspond to the positions of the variables that have been eliminated from the previous equations. Therefore, we have (possibly with an initial reordering of the equations)

$$j_0 < j_1 < \dots < j_{l-1} < n, \tag{18}$$

and the rows l to $s-1$ are all zero. That is, at least one of the variables is eliminated at each step (and possibly more than one). Note that, after the lth equation, all the variables are eliminated. If $\tilde{b}_l = \dots = \tilde{b}_{s-1} = 0$, $\operatorname{rank}(\tilde{\mathbf{A}} \mid \tilde{b}) = \operatorname{rank}(\tilde{\mathbf{A}}) = l$ and the system admits a solution. The solution of the system is obtained by back substitution.

The procedure of Gaussian elimination can be extended to the case of polynomial equations. This extension is known as Buchberger's algorithm and the set of equations obtained after elimination is called a Gröbner basis. In order to give an overview of the algorithm, we recall the theoretical background and show the analogy with Gaussian elimination. We refer to the bibliography for the details and formal proofs.

As in the linear case, we need to define an ordering of the terms of (12), i.e. the monomials of $x_0, \dots, x_{n-1}$. Since the variables may appear with different exponents, there are different ways to order monomials according to the variables and the exponents. A common choice is *lexicographic* (*lex*) *ordering*.

Definition 3.2 *(Lexicographic ordering).* Let $\mathbf{d} = (d_0, \dots, d_{n-1})$ and $\mathbf{d}' = (d'_0, \dots, d'_{n-1})$ be two n-tuples representing positive integer exponents of the monomials $\mathbf{x}^{\mathbf{d}}$, $\mathbf{x}^{\mathbf{d}'}$. We say that $\mathbf{d} >_{\text{lex}} \mathbf{d}'$ if, in the vector difference $\mathbf{d} - \mathbf{d}' \in \mathbb{Z}^n$, the left-most nonzero entry is positive. We will write $\mathbf{x}^{\mathbf{d}} >_{\text{lex}} \mathbf{x}^{\mathbf{d}'}$ if $\mathbf{d} >_{\text{lex}} \mathbf{d}'$.

Note that, next to the type of ordering, we also need to define the order between the different variables. In the following, we will assume that the terms of each polynomial are ordered in descending order according to lex ordering, and with $x_0 > x_1 > \cdots > x_{n-1}$. We define the *multidegree* of a polynomial p, multideg(p) as the largest exponent of the monomials of p according to the lex ordering. We call *leading term*, LT(p) the term of p with the largest exponent. The *total degree* of a polynomial is defined as the maximum sum of the exponent vectors $\mathbf{d}$ of its terms.

Example 3.1. Let us consider a polynomial

$$p = 2x_0^3x_1^2 + 5x_0x_1^3x_2^3 + 3x_1^4x_2. \tag{19}$$

Using lex ordering, and $x_0 > x_1 > x_2$, we have $x_0^3x_1^2 > x_0x_1^3x_2^3 > x_1^4x_2$, and (19) is ordered in descending lexicographic order. Its multidegree is multideg$(p) = (3, 2, 0)$, and the leading term LT$(p) = 2x_0^3x_1^2$. The total degree is $1 + 3 + 3 = 7$.

In the procedure of Gaussian elimination, the equations of the system correspond to a set of vectors generating a subspace. The aim of elimination is to determine a new basis for such a subspace with the structure given by (16). In the case of polynomials, the equations can be combined using polynomial coefficients. The set of all polynomials that can be constructed from an original set has the algebraic structure of an ideal of the ring of polynomials.

Definition 3.3 *(Ideal).* A subset $I \subset \mathbb{C}[x_0, \dots, x_{n-1}]$ is an *ideal* if it satisfies:

(1) $0 \in I$.
(2) If $p, q \in I$, then $p + q \in I$.
(3) If $p \in I$ and $a \in \mathbb{C}[x_0, \dots, x_{n-1}]$, then $ap \in I$.

If $p_0, \dots, p_{s-1}$ are polynomials, then we set

$$I = \langle p_0, \dots, p_{s-1} \rangle = \left\{ \sum_{i=0}^{s-1} a_i p_i \colon a_i \in \mathbb{C}[x_0, \dots, x_{n-1}] \right\}. \tag{20}$$

We call I the ideal generated by $p_0, \dots, p_{s-1}$.

3.2. The ideal membership problem

A key problem in algebra is to determine whether a given element p of a ring belongs to a given ideal I or not. In terms of polynomials, the problem is equivalent to testing if a given polynomial p can be written as a linear combination of the polynomial generators of I, $p_0, \dots, p_{s-1}$, using polynomial coefficients $a_0, \dots, a_{s-1}$. Such a problem is known as the *ideal membership problem*.

If we think of an ideal generated by a single polynomial in one variable, the problem has a simple solution. In fact, we can apply the algorithm of polynomial division and write p as

$$p = a_0 p_0 + r. \tag{21}$$

The quotient a_0 and the remainder r are uniquely determined under the condition that $\deg(r) < \deg(p_0)$. In this case, the ideal membership problem has a simple solution: if $r = 0$, p belongs to $\langle p_0 \rangle$, otherwise not.

In the case of multiple polynomials in multiple variables, we can extend the algorithm of polynomial division. The goal is to write p as

$$p = a_0 p_0 + \cdots + a_{s-1} p_{s-1} + r. \tag{22}$$

The division algorithm consists in considering the monomials of p in decreasing order. For each monomial, if the leading term of one of the p_i's is a divisor, then the corresponding quotient a_i is updated together with the remaining monomials of p. Otherwise, the monomial is moved to the remainder r. The following theorem can be proven for polynomial division [5, §2.3, Theorem 3].

Theorem 3.1. *Fix a monomial order and let $P = (p_0, \dots, p_{s-1})$ be an ordered s-tuple of polynomials in $x_0, \dots, x_{n-1}$. Then every polynomial p can be written as in* (22)*, where either $r = 0$ or r is a linear combination of monomials, none of which is divisible by any of* $\mathrm{LT}(p_0), \dots, \mathrm{LT}(p_{s-1})$*. Furthermore, we have*

$$\operatorname{multideg}(p) \geqslant \operatorname{multideg}(a_i p_i), \quad i = 0, \dots, s-1. \tag{23}$$

A crucial point of the algorithm is that the result of the division depends on the order that we consider for the divisors $p_0, \dots, p_{s-1}$.

Example 3.2. Let $p_0 = x_0 x_1 + 1$, $p_1 = x_1^2 - 1$ be two polynomials in x_0, x_1 and assume we use the lex order with $x_0 > x_1$. If we divide $p = x_0 x_1^2 - x_0$ by $P = (p_0, p_1)$ the result is

$$x_0 x_1^2 - x_0 = x_1 \cdot (x_0 x_1 + 1) + 0 \cdot \left(x_1^2 - 1\right) + (-x_0 - x_1). \tag{24}$$

With $P = (p_1, p_0)$, however, we have

$$x_0 x_1^2 - x_0 = x_0 \cdot \left(x_1^2 - 1\right) + 0 \cdot (x_0 x_1 + 1) + 0. \tag{25}$$

Therefore, the result of division is not unique. Moreover, the remainder of division may be nonzero, even if $p \in \langle p_0, p_1 \rangle$. In the following, we will denote $\overline{p}^P$ the remainder r of the division of p by the s-tuple of polynomials P.

There are some cases where the s-tuple of polynomials has a particular structure that allows to solve the ambiguity. A set with such a property is called a Gröbner basis.

Definition 3.4 *(Gröbner basis).* Let $G = \{g_0, \dots, g_{u-1}\}$ be a basis for the ideal I. If for all $p \in I$ the remainder of the division $\overline{p}^G = 0$ then G is called a *Gröbner basis* for I.

Gröbner bases have several interesting properties, including a generalization of the structure of the system (16). However, the most surprising result is given by the following theorem [4, §2.5, Theorem 4]:

Theorem 3.2 *(Hilbert basis theorem). Every ideal I of the ring of polynomials of n variables has a finite generating set. That is, $I = \langle g_0, \dots, g_{u-1} \rangle$ for some $g_0, \dots, g_{u-1} \in I$. In particular, it is always possible to choose $g_0, \dots, g_{u-1}$ so that they form a Gröbner basis.*

3.3. Buchberger's algorithm

The key step of Gaussian elimination was to combine two rows of the matrix (i.e. two equations) in order to cancel the entry corresponding to the variable of highest order. This concept is extended to polynomials by introducing S-polynomials.

Definition 3.5 *(S-polynomial).* Let p_0, p_1 be two nonzero polynomials in $x_0, \dots, x_{n-1}$. If $\operatorname{multideg}(p_0) = \mathbf{d}$ and $\operatorname{multideg}(p_1) = \mathbf{d}'$, then let $\mathbf{d}'' = (d_0'', \dots, d_{n-1}'')$, where $d_i'' = \max(d_i, d_i')$. The *S-polynomial of p_0 and p_1* is defined as the linear combination

$$S(p_0, p_1) = \frac{\mathbf{x}^{\mathbf{d}''}}{\mathrm{LT}(p_0)} p_0 - \frac{\mathbf{x}^{\mathbf{d}''}}{\mathrm{LT}(p_1)} p_1. \tag{26}$$

Algorithm 1. Buchberger's algorithm for the computation of a Gröbner basis.

Let $I = \langle p_0, \ldots, p_{s-1} \rangle \neq 0$ be a polynomial ideal. Then a Gröbner basis for I can be constructed in a finite number of steps by the following algorithm:

Input: $P = (p_0, \ldots, p_{s-1})$
Output: a Gröbner basis $G = (g_0, \ldots, g_{u-1})$ for I, with $P \subseteq G$
$G := P$
Repeat
  $G' := G$
  For each pair (p, q), $p \neq q$ in G' do
    $S := \overline{S(p, q)}^{G'}$
    If $S \neq 0$ then $G := G \cup S$
until $G = G'$.

Using S-polynomials, we can easily verify if a basis G is a Gröbner basis. In fact, we have the following theorem [5, §2.6, Theorem 6]:

Theorem 3.3. *Let I be a polynomial ideal. Then a basis $G = \{g_0, \ldots, g_{u-1}\}$ is a Gröbner basis for I if and only if, for all pairs $i \neq j$, the remainder on division of $S(g_i, g_j)$ by G (listed in some order) is zero.*

There is a main difference between the linear and the polynomial case when we combine equations. In the linear case, if we combine p_0 and p_1 we obtain an equation of the form

$$h = ap_0 + bp_1, \quad a, b \in \mathbb{C}, \tag{27}$$

and this equation can be used to replace p_0 or p_1, i.e.

$$\langle p_0, p_1 \rangle = \langle p_0, h \rangle = \langle h, p_1 \rangle. \tag{28}$$

In the polynomial case, equations are combined using polynomial coefficients, i.e. the terms a and b are polynomials in the variables $x_0, \ldots, x_{n-1}$. Since the set of polynomials has the structure of a ring, it is not always possible to find an inverse of the coefficients. This means that, for example, it is not always possible to compute p_1 from p_0 and h. For this reason, to construct a Gröbner basis, one has to increase initially the number of elements of the basis. Such an extension ends when the conditions given by Theorem 3.3 are satisfied. This algorithm is due to Buchberger and is given in Algorithm 1 [2, §2.7, Theorem 2].

Algorithm 1 is not a very practical way to compute a Gröbner basis. Several improvements are possible. Moreover, Gröbner bases computed in this way are often bigger than necessary. For this reason, unneeded generators are eliminated by using Theorem 3.3 or similar tests.

3.4. Solution of polynomial equations

We can now show that a Gröbner basis corresponding to a system of polynomial equations and built using lex ordering simplifies the system and allows to compute the solution by back substitution. Remember that we defined the ideal I as the set of all polynomials that can be derived from the initial set using polynomial coefficients. We can also define the *elimination ideal* I_k as the set of all polynomials that can be deduced from the original system and contain only the variables $x_k, \ldots, x_{n-1}$,

$$I_k = I \cap \mathbb{C}[x_k, \ldots, x_{n-1}]. \tag{29}$$

If we can find a basis for each one of the sets I_k, $k = 1, \ldots, n-1$, we can determine the solutions of the original system using back substitution. In fact, we clearly have that for any $k \geqslant 1$, $I_{k+1} \subseteq I_k$. Therefore, if we have a solution of the system of equations associated to I_{k+1}, we can extend it to the system associated to I_k by computing the values of the variable x_k. This can be done by computing the zeros of a polynomial in the variable x_k. An important property

Algorithm 2. Algorithm for multichannel sampling with unknown offsets using Gröbner bases.

(1) Write out the equations from (5) describing the samples as a function of the signal coefficients.
(2) If necessary, perform a change of variables to convert the equations into a set of polynomial equations.
(3) Compute a Gröbner basis for the set of polynomial equations using Buchberger's algorithm.
(4) Use back substitution to compute the offsets and signal parameters from the Gröbner basis.
(5) If necessary, eliminate solutions that are not valid (e.g. offset values not on the unit circle in the Fourier case).

of Gröbner bases is that they solve easily the problem of determining the ideals I_k, $k = 1, \dots, n-1$. Namely, the Gröbner bases of all the ideals I_k, $k = 1, \dots, n-1$ can be determined from the Gröbner basis of I. The result is given by the elimination theorem [2, §3.1, Theorem 2]:

Theorem 3.4 *(Elimination theorem). Let $I \subset [x_0, \dots, x_{n-1}]$ be an ideal and let G be a Gröbner basis of I with respect to lex order where $x_0 > x_1 > \dots > x_{n-1}$. Then, for every $1 \leqslant k < n$, the set*

$$G_k = G \cap \mathbb{C}[x_k, \dots, x_{n-1}] \tag{30}$$

is a Gröbner basis of the kth elimination ideal I_k.

Using this theorem, we can compute the different variables from a Gröbner basis using back substitution. To summarize, we can solve a set of polynomial equations in multiple variables as follows. First, we compute a Gröbner basis for the ideal corresponding to the set of equations using Buchberger's algorithm. The solution can then be obtained from this Gröbner basis using back substitution.

4. Multichannel sampling using Gröbner bases

We can now use Gröbner bases and Buchberger's algorithm to solve the equations from (5). After a possible change of variables to write the equations in polynomial form, we can directly apply Buchberger's algorithm. This results in a Gröbner basis for the ideal defined by the set of equations. The signal parameters can then be easily extracted from this Gröbner basis using the elimination theorem. This is summarized in Algorithm 2. We will illustrate this algorithm with two examples, for polynomial signals and signals described by Fourier series, respectively.

Example 4.1 *(Polynomial signals).* First, we reconsider the equations obtained in Example 2.1. That is, we consider a second degree polynomial signal with two sets of two samples ($L = 3$, $M = 2$, and $N = 2$, see also Fig. 2). We can represent the set of solutions of (6) as the points of the affine variety defined by the set of polynomials:

$$\begin{aligned}
p_0 &= \alpha_2 + 4, \\
p_1 &= \frac{1}{4}\alpha_0 + \frac{1}{2}\alpha_1 + \alpha_2, \\
p_2 &= \frac{1}{4}\alpha_0 t_1^2 + \frac{1}{2}\alpha_1 t_1 + \alpha_2 + 6, \\
p_3 &= \frac{1}{4}\alpha_0 t_1^2 + \frac{1}{2}\alpha_0 t_1 + \frac{1}{4}\alpha_0 + \frac{1}{2}\alpha_1 t_1 + \frac{1}{2}\alpha_1 + \alpha_2 - 6,
\end{aligned} \tag{31}$$

in the variables α_0, α_1, α_2 and t_1. We fix the ordering of variables as $\alpha_0 > \alpha_1 > \alpha_2 > t_1$ and we use lex ordering for monomials.

At the first step of Buchberger's algorithm, we find that

$$S(p_0, p_1) = 4\alpha_0 - 2\alpha_1\alpha_2 - 4\alpha_2^2 = (-2\alpha_1 - 4\alpha_2)p_0 + 16p_1,$$

$$\begin{aligned}
S(p_0, p_2) &= \alpha_0 t_1^2 - \frac{1}{2}\alpha_1\alpha_2 t_1 - \alpha_2^2 - 6\alpha_2 \\
&= \left(-\frac{1}{2}\alpha_1 t_1 - \alpha_2 - 4t_1^2 - 2\right)p_0 + 4t_1^2 p_1 - 2\alpha_1 t_1^2 + 2\alpha_1 t_1 + 16t_1^2 + 8, \qquad (32)\\
S(p_0, p_3) &= -\frac{1}{2}\alpha_0\alpha_2 t_1 - \frac{1}{4}\alpha_0\alpha_2 + \alpha_0 t_1^2 - \frac{1}{2}\alpha_1\alpha_2 t_1 - \frac{1}{2}\alpha_1\alpha_2 - \alpha_2^2 + 6\alpha_2 \\
&= \left(-\frac{1}{2}\alpha_0 t_1 - \frac{1}{4}\alpha_0 - \frac{1}{2}\alpha_1 t_1 - \frac{1}{2}\alpha_1 - \alpha_2 - 4t_1^2 - 8t_1 + 6\right)p_0 \\
&\quad + \left(4t_1^2 + 8t_1 + 4\right)p_1 - 2\alpha_1 t_1^2 - 2\alpha_1 t_1 + 16t_1^2 + 32t_1 - 24, \\
S(p_1, p_2) &= \frac{1}{8}\alpha_1 t_1^2 - \frac{1}{8}\alpha_1 t_1 + \frac{1}{4}\alpha_2 t_1^2 - \frac{1}{4}\alpha_2 - \frac{3}{2} \\
&= \left(\frac{1}{4}t_1^2 - \frac{1}{4}\right)p_0 + \frac{1}{8}\alpha_1 t_1^2 - \frac{1}{8}\alpha_1 t_1 - t_1^2 - \frac{1}{2}, \\
S(p_1, p_3) &= -\frac{1}{8}\alpha_0 t_1 - \frac{1}{16}\alpha_0 + \frac{1}{8}\alpha_1 t_1^2 - \frac{1}{8}\alpha_1 t_1 - \frac{1}{8}\alpha_1 + \frac{1}{4}\alpha_2 t_1^2 - \frac{1}{4}\alpha_2 + \frac{3}{2} \\
&= \left(\frac{1}{4}t_1^2 + \frac{1}{2}t_1\right)p_0 + \left(-\frac{1}{2}t_1 - \frac{1}{4}\right)p_1 + \frac{1}{8}\alpha_1 t_1^2 + \frac{1}{8}\alpha_1 t_1 - t_1^2 - 2t_1 + \frac{3}{2}, \\
S(p_2, p_3) &= -\frac{1}{2}\alpha_0 t_1 - \frac{1}{4}\alpha_0 - \frac{1}{2}\alpha_1 + 12 = (2t_1 + 1)p_0 + (-2t_1 - 1)p_1 + \alpha_1 t_1 - 8t_1 + 8. \qquad (33)
\end{aligned}$$

Therefore, we add the remainders that are nonzero to the basis:

$$\begin{aligned}
p_4 &= \overline{S(p_0, p_2)}^G = -2\alpha_1 t_1^2 + 2\alpha_1 t_1 + 16t_1^2 + 8, \\
p_5 &= \overline{S(p_0, p_3)}^G = -2\alpha_1 t_1^2 - 2\alpha_1 t_1 + 16t_1^2 + 32t_1 - 24, \\
p_6 &= \overline{S(p_2, p_3)}^G = \alpha_1 t_1 - 8t_1 + 8. \qquad (34)
\end{aligned}$$

The remainders of $S(p_1, p_2)$ and $S(p_1, p_3)$ are not added, because they are the same as polynomials p_4 and p_5, respectively. Following the same procedure, in the second iteration, we find that only $S(p_2, p_6)$ and $S(p_4, p_6)$ give a distinct, nonzero remainder. We add the polynomials

$$p_7 = \overline{S(p_2, p_6)}^G = -2\alpha_1 - 48, \qquad p_8 = \overline{S(p_4, p_6)}^G = 32t_1 - 8 \qquad (35)$$

to the basis. In the following iteration all remainders are zero and by Theorem 3.3 we conclude that $p_0, \dots, p_8$ is a Gröbner basis. Applying again Theorem 3.3 we can try to reduce the elements of the basis. In this case, we have that p_2, p_3, p_4, p_5, p_6 can be removed and the final basis is given by $\{p_0, p_1, p_7, p_8\}$. In order to apply the elimination theorem, we rename the elements of the basis as:

$$g_0 = \frac{1}{4}\alpha_0 + \frac{1}{2}\alpha_1 + \alpha_2, \qquad g_1 = -2\alpha_1 - 48, \qquad g_2 = \alpha_2 + 4, \qquad g_3 = 32t_1 - 8. \qquad (36)$$

The elimination ideals are $I_1 = \langle g_1, g_2, g_3\rangle$, $I_2 = \langle g_2, g_3\rangle$, and $I_3 = \langle g_3\rangle$. The solution of the problem can be obtained by computing the points of the affine variety associated to I_3 and extending it by back substitution to I_2, I_1 and I. We easily find that the unique solution is given by $t_1 = \frac{1}{4}$, $\alpha_2 = -4$, $\alpha_1 = -24$, and $\alpha_0 = 64$.

The procedure described in the above example can be applied to any multichannel sampling problem in the polynomial space $\mathcal{H}$. For any value of the variables L, M, and N, the equations in (5) form a set of polynomial equations and we can therefore compute the parameter values by calculating a Gröbner basis for the corresponding ideal. Similarly, the same algorithm can be applied to Fourier series, using the change of variables given in Section 2. This is a very interesting case from a practical point of view, as signals and images are often bandlimited or can be considered to be so.

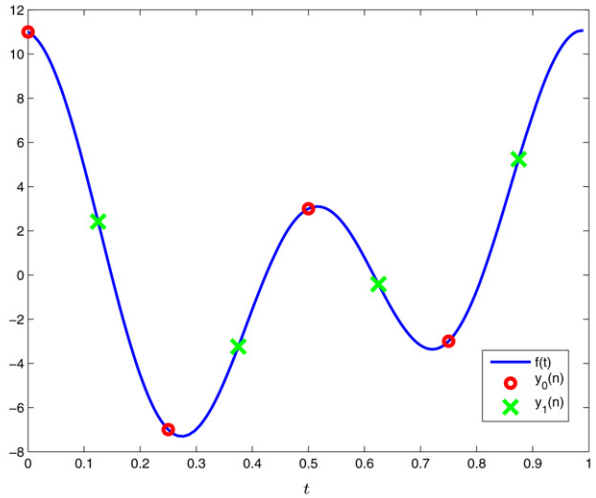

Fig. 3. Fourier series signal used in Example 4.2. The signal with Fourier series coefficients $\mathbf{x} = (3\ \ 2-j\ \ 1\ \ 2+j\ \ 3)^T$ is sampled with two sets of four samples $\mathbf{y}_0 = (11\ \ -7\ \ 3\ \ -3)^T$ and $\mathbf{y}_1 = (1+\sqrt{2}\ \ 1-3\sqrt{2}\ \ 1-\sqrt{2}\ \ 1+3\sqrt{2})^T$ with offset $t_1 = 1/2$.

Example 4.2 *(Fourier series).* Assume $K = 2$ (and therefore $L = 5$, see (7)), i.e. the input signal is represented by the parameter vector $\boldsymbol{\alpha} = (\alpha_{-2}, \dots, \alpha_2)$, where each entry is a complex value. For this example, we assume

$$\boldsymbol{\alpha} = (3 \quad 2-j \quad 1 \quad 2+j \quad 3)^T.$$

We suppose that $M = 2$ sets of $N = 4$ samples are taken from the input signal, with the displacements $\mathbf{t} = (0, 1/2)$. In this case, the two sets of measurements are

$$\begin{aligned} \mathbf{y}_0 &= (11 \quad -7 \quad 3 \quad -3)^T, \\ \mathbf{y}_1 &= (1+\sqrt{2} \quad 1-3\sqrt{2} \quad 1-\sqrt{2} \quad 1+3\sqrt{2})^T. \end{aligned} \tag{37}$$

The signal and its samples are shown in Fig. 3. Applying (11), we obtain 8 polynomials that represent the constraints imposed by the measurements:

$$\begin{aligned} p_0 &= \alpha_2 + \alpha_1 + \alpha_0 + \alpha_{-1} + \alpha_{-2} - 11, \\ p_1 &= -\alpha_2 + j\alpha_1 + \alpha_0 - j\alpha_{-1} - \alpha_{-2} + 7, \\ p_2 &= \alpha_2 - \alpha_1 + \alpha_0 - \alpha_{-1} + \alpha_{-2} - 3, \\ p_3 &= -\alpha_2 - j\alpha_1 + \alpha_0 + j\alpha_{-1} - \alpha_{-2} + 3, \\ p_4 &= \alpha_2 z_1^4 + \alpha_1 z_1^3 + \alpha_0 z_1^2 + \alpha_{-1} z_1 + \alpha_{-2} - (1+\sqrt{2}) z_1^2, \\ p_5 &= -\alpha_2 z_1^4 + j\alpha_1 z_1^3 + \alpha_0 z_1^2 - j\alpha_{-1} z_1 - \alpha_{-2} - (1-3\sqrt{2}) z_1^2, \\ p_6 &= \alpha_2 z_1^4 - \alpha_1 z_1^3 + \alpha_0 z_1^2 - \alpha_{-1} z_1 + \alpha_{-2} - (1-\sqrt{2}) z_1^2, \\ p_7 &= -\alpha_2 z_1^4 - j\alpha_1 z_1^3 + \alpha_0 z_1^2 + j\alpha_{-1} z_1 - \alpha_{-2} - (1+3\sqrt{2}) z_1^2, \end{aligned} \tag{38}$$

where the complex variable $z_1 = e^{j2\pi t_1/4}$ represents the displacement. Again, by using Buchberger's algorithm, we obtain a Gröbner basis. Assuming the ordering $\alpha_2 > \alpha_1 > \cdots > \alpha_{-2} > z_1$, we obtain

$$\begin{aligned} g_0 &= 2\alpha_2 - 3j\sqrt{2} z_1 + 3\sqrt{2} z_1 - 12, \\ g_1 &= \alpha_1 - 2 - j, \\ g_2 &= \alpha_0 - 1, \\ g_3 &= \alpha_{-1} - 2 + j, \\ g_4 &= 2\alpha_{-2} + 3j\sqrt{2} z_1 - 3\sqrt{2} z_1, \\ g_5 &= 2z_1^2 - \sqrt{2}(1+j) z_1 = 2z_1\left(z_1 - \frac{\sqrt{2}}{2}(1+j)\right). \end{aligned} \tag{39}$$

Algorithm 3. Algorithm for multichannel sampling from noisy samples.

(1) Write out the equations from (5) describing the samples as a function of the signal coefficients.
(2) If necessary, perform a change of variables to convert the equations into a set of polynomial equations.
(3) Divide these equations into at most $\binom{MN}{L+M-1}$ critical subsets of equations $\mathcal{S}_i$.
(4) Compute a Gröbner basis for each set $\mathcal{S}_i$. Use back substitution to obtain the offsets and the signal parameters.
(5) Eliminate solutions that are not valid (e.g. offset values not on the unit circle in the Fourier case).
(6) Compute the weighted average of the offsets corresponding to the remaining solutions (typically one per set $\mathcal{S}_i$).
(7) Fill in the offsets in the equations from (5) and solve the set of linear equations for the unknown signal parameters.

In the last polynomial of the basis, g_5, all variables but z_1 are eliminated. Therefore we can compute the solutions for the displacement variable, $z_1 = 0$ and $z_1 = e^{j\pi/4}$, from g_5. Clearly, $z_1 = 0$ is discarded since it does not belong to the unit circle, while the second solution corresponds to the correct displacement $t_1 = 1/2$. By back substitution, one can compute the signal parameters.

To sum up, in the above examples we have MN polynomial equations with maximum total degree L. The equations are linear in the signal coefficients α_l, and polynomial of order at most $L-1$ in the offsets t_m. The computed Gröbner basis is linear in Example 4.1, and contains a second degree polynomial in Example 4.2. This is much lower than the theoretical double exponential bound that will be discussed in Section 6.

5. Multichannel sampling under noisy conditions

The computation of a Gröbner basis is typically performed with infinite precision. A Gröbner basis is defined as a set of polynomials that generates the same variety as the original set of polynomial equations. The solution that is computed using Gröbner bases is therefore an exact solution to the set of polynomial equations.

Moreover, concepts such as projections or distance do not have any meaning over the ring of polynomials. It is not possible to compute a 'least squares solution' to a set of equations with Gröbner bases. Hence, if the measurements are noisy, or known with limited precision, Buchberger's algorithm would generally conclude that there is no solution. As there are usually more equations than unknowns (see Example 4.2), the errors on the sample values make the equations from (5) incoherent. There has already been a lot of research on the stability of Gröbner basis computation, and various solutions have been proposed [20–22].

We propose to solve this problem by dividing the complete set of polynomial equations into multiple (overlapping) critical subsets. By critical we mean that there is a finite, nonempty set of solutions (typically when the number of equations is equal to the number of unknowns). We could use all the critical subsets that can be derived from the original set of equations, or select only a limited number of them to limit the computational time. We can now compute a Gröbner basis for each subset, and obtain a set of parameter values using back substitution. The final solution can then be defined as a (weighted) average of the different solutions from the subsets. This method is summarized in Algorithm 3. Let us now analyze an example.

Example 5.1 *(Fourier series with noisy measurements).* Consider a signal that is represented by its $L = 5$ Fourier series coefficients, given by

$$\boldsymbol{\alpha} = (\,5-j \quad -3j \quad -6 \quad 3j \quad 5+j\,)^T. \tag{40}$$

The signal is sampled with two sets of four samples ($M = 2$, $N = 4$), with an offset vector $\mathbf{t} = (0, 24/11) = (0, 2.1818)$. In a noiseless case, this would result in the following two sets of samples:

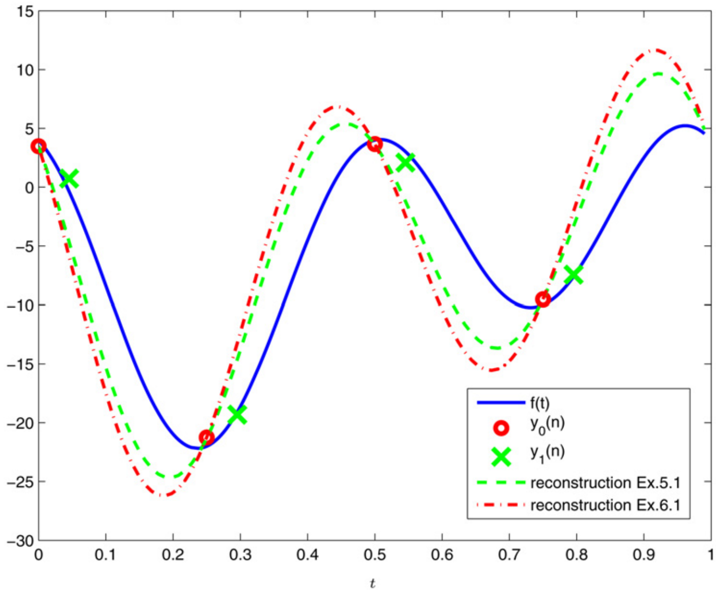

Fig. 4. Fourier series signal used in Examples 5.1 and 6.1. The signal with Fourier series coefficients $\mathbf{x} = (5 - j\ \ -3j\ \ -6\ \ 3j\ \ 5 + j)^T$ is sampled with two sets of four noisy samples $\mathbf{y}_0 = (3.4845\ \ -21.2468\ \ 3.6672\ \ -9.5310)^T$ and $\mathbf{y}_1 = (2.0917\ \ -7.4480\ \ 0.7300\ \ -19.3078)^T$ with offset $t_1 = 24/11$. Its reconstruction both in Example 5.1 (−−) and Example 6.1 (− · −) are shown. We can see that the results are rather unstable even with small amounts of noise. Recall however that the sampling locations of the second set (x) with respect to the first set are unknown.

$$\begin{aligned} \mathbf{y}_0 &= (\,4 \quad -22 \quad 4 \quad -10\,)^T, \\ \mathbf{y}_1 &= (\,3.0217 \quad -7.5743 \quad -0.3591 \quad -19.0882\,)^T. \end{aligned} \tag{41}$$

The second set of samples is given numerically, because the exact expressions are quite complicated. Now we add white Gaussian noise to these samples with mean 0 and standard deviation 1, resulting in the noisy sample values

$$\begin{aligned} \mathbf{y}_0 &= (\,3.4845 \quad -21.2468 \quad 3.6672 \quad -9.5310\,)^T, \\ \mathbf{y}_1 &= (\,2.0917 \quad -7.4480 \quad 0.7300 \quad -19.3078\,)^T, \end{aligned} \tag{42}$$

for one particular realization (see also Fig. 4). We obtain a similar set of polynomials as in (38), with just different sample values. As we have 8 equations in 6 unknowns (5 signal parameters and an offset), we compute a Gröbner basis for all $\binom{8}{6} = 28$ subsets $\mathcal{S}_i$ of 6 polynomials from the total set. One of them is given here:

$$\begin{aligned} g_0 &= \alpha_2 - (11.5043 + 8.2663j)z_1 - (8.5363 + 13.2582j)z_1^2 - 9.4824, \\ g_1 &= 0.04567 - 2.9289j + \alpha_1, \\ g_2 &= 5.9065 + \alpha_0, \\ g_3 &= 0.04567 + 2.9289j + \alpha_{-1}, \\ g_4 &= \alpha_{-2} + (11.5043 + 8.2663j)z_1 + (8.5363 + 13.2582j)z_1^2, \\ g_5 &= z_1^3 + (1.1192 + 1.0848j)z_1^2 + (0.0312 + 0.9995j)z_1. \end{aligned} \tag{43}$$

We can then compute all the possible solutions for each of the Gröbner bases. We eliminate the invalid ones: those that do not correspond to valid offsets (values of z_1 that are not on the unit circle), as well as those that give a large error when evaluated on the two remaining equations. Typically, only a single solution remains for every Gröbner basis. From the remaining solutions, we compute the offsets t_1, and compute their average value:

$$t_{1,\text{avg}} = 2.0660. \tag{44}$$

This way of proceeding has the advantage that we keep a valid offset value. If we would just average the computed values for z_1, the result is typically not on the unit circle anymore, and does not represent a valid offset. Note that we performed a simple averaging operation here. A weighted average that takes the sensitivity of the results to the different sample values into account would probably improve the results further. We replace this average offset value in the original equations, and compute the least squares solution of this set of linear equations in the unknown signal parameters $\boldsymbol{\alpha}$:

$$\hat{\boldsymbol{\alpha}} = \begin{pmatrix} 4.7412 - 4.5812j \\ -0.0388 - 2.9566j \\ -5.9450 \\ -0.0388 + 2.9566j \\ 4.7412 + 4.5812j \end{pmatrix}. \tag{45}$$

The relative error is computed as the norm of the difference between the true coefficient vector $\boldsymbol{\alpha}$ and the estimated coefficient vector $\hat{\boldsymbol{\alpha}}$ divided by the norm of the coefficient vector: $\|\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}\|/\|\boldsymbol{\alpha}\|$. For this simulation, we obtain a relative error of 0.493. This error can be compared to the error that would be obtained from the noisy samples with the exact offset t_1, which is 0.080. Averaged over 250 such simulations with random signal coefficients and offsets, the estimated relative error is 0.340, compared to 0.095 in the ideal case using the exact offsets with the noisy samples.

6. Complexity and optimizations

The main disadvantage of Gröbner bases for the multichannel sampling problem is the computational complexity of Buchberger's algorithm. As explained in Section 3, the set of polynomials p_i has to be expanded in the first part of the algorithm by adding the nonzero remainders of S-polynomials. Unlike in Gaussian elimination, we cannot simply replace a polynomial by a linear combination of that polynomial with another one. The linear combination has to be added to the existing set of polynomials. This expansion can become very large, and is one of the reasons for the high memory requirements of Buchberger's algorithm. The maximum total degree of the polynomials in a reduced Gröbner basis can be shown to be

$$E = 2\left(\frac{D^2}{2} + D\right)^{2^{S-1}}, \tag{46}$$

where D is the maximal total degree of the polynomials p_i, and S is the number of variables [7,15]. In our setup, the maximal total degree $D = L$, and we would therefore typically obtain

$$E = 2\left(\frac{L^2}{2} + L\right)^{2^{L+M-2}}. \tag{47}$$

Fortunately, this double exponential function describes a worst-case scenario, while in practice the complexity is often much lower. In the examples from the previous sections, the degree was always much lower, with only linear terms remaining in Example 4.1, and a second degree polynomial in Example 4.2. The above upper bounds for those cases would be about 10^7 and 10^{40}, respectively.

Another reason for the high complexity of Buchberger's algorithm is given by the fact that the algorithm performs computations with infinite precision. If for example the input coefficients are (small) integers, quite complicated rational numbers are used in the computation of a Gröbner basis. Using Maple to solve a polynomial problem like the one in Example 2.1, with a 6th degree polynomial, and 3 sets of 3 samples, the algorithm already requires more than 1 GB of memory. In the back substitution step of our solution method, we need to compute the zeros of a polynomial. The complexity of this operation will depend on the order of the specific polynomial that is obtained. Although theoretically, this order can only be bounded by (47), in practice, it is often much lower (as can also be seen from the examples). The roots of a polynomial with degree E can be computed using an algorithm with complexity

$$\left(E(\log E)^2|\log\epsilon| + E^2(\log E)^2\right) \tag{48}$$

where ϵ is the precision of the computed roots [13].

Various optimizations of Buchberger's algorithm exist. For example, certain S-polynomials can already be excluded before examining them. Often, other orderings than the lexicographic ordering also result in lower complexity. Algorithms exist to convert a Gröbner basis using one ordering into a Gröbner basis for another ordering. It can therefore be computationally more efficient to compute a Gröbner basis first using another ordering, and convert it then into lexicographic ordering. Lexicographic ordering is required to apply the elimination theorem, which offers a simple way to compute the coefficients using back substitution. Various implementations of Gröbner basis algorithms including different optimizations exist (Gb [9], Macaulay2, Maple, Mathematica, Magma, Singular). We used Mathematica for our simulations. Even though this is probably not the optimal implementation [9], it allows us to implement and clearly show all the important concepts and ideas from this paper.

It is important to note that the multichannel sampling problem has a particular structure. From (5), which describes the problem for any kind of basis, and from the different examples in previous sections, we can see that the equations are linear in the signal parameters. They only have higher polynomial orders in the offset parameters $\mathbf{t}$. Typically, there are many (L) signal parameters, while only a small number (M) of different sets of samples is used. The (linear) signal parameters can be eliminated from the set of equations using Gaussian elimination on the first L equations. This can be performed in $O(L^3)$ operations, and for our particular structure of the problem and with two sets of samples ($M = 2$), it does not increase the degree of the polynomial coefficients in $\mathbf{t}$. This can be seen from the examples in the previous sections. With a Fourier basis, each term in a signal parameter α_i has the same power of the offset variable z_1, and a varying complex coefficient (or has no offset variable at all, for the first set of N equations). The signal parameters can therefore be eliminated by multiplying equations with complex numbers and adding them together. We never need to multiply any of the equations from the second set by the offset variable z_1, and therefore do not increase its degree. For polynomial signals, we can perform a similar elimination. By ordering the signal parameters as $\alpha_{L-1}, \alpha_{L-2}, \dots, \alpha_0$, we can eliminate each of the parameters without needing to multiply equations by the offset variable t_1. If more than $M = 2$ sets are considered, the different offset variables have to be multiplied in the Gaussian elimination, and the results are more complex.

After this Gaussian elimination step, the computed values for the signal parameters (as a function of $\mathbf{t}$) can be replaced in the $MN - L$ remaining equations. We obtain a (much smaller) set of $MN - L$ polynomial equations in the unknown offsets $\mathbf{t}$. It is now sufficient to compute a Gröbner basis for this smaller set in much fewer unknowns ($M \ll L$). With noisy samples, we can now compute Gröbner bases for the $\binom{MN-L}{M-1}$ subsets of $M-1$ equations instead of the $\binom{MN}{L+M-1}$ sets of $L + M - 1$ equations previously. Typically this results in much fewer subsets of smaller size. However, the precision with which the parameters are computed is also (slightly) lower. Compared to all the possible subsets of $L + M - 1$ equations in Algorithm 3, now only the subsets containing the first L equations and all possible combinations of $M - 1$ equations from the remaining set are considered. The maximum total degree of a Gröbner basis for such a subset is reduced to

$$2\left(\frac{L^2}{2} + L\right)^{2^{M-2}}, \tag{49}$$

where the number of sets of samples M is much smaller than the number of coefficients L. As the first L equations are linear in the signal parameters and have lower degrees than (49) in the offsets, this bound also replaces the previous bound (47) for the general Gröbner basis computation. For Examples 4.1 and 4.2, these bounds are 15 and 35, respectively. While this is still far beyond the actual degrees of the Gröbner bases, it is already a much tighter bound than the one given in (47). Once this (smaller) Gröbner basis is computed, the offset values can be obtained using back substitution and a method to compute the zeros of a polynomial. We can compute the signal parameters by substituting the offset values in the first L equations. Once the offsets are known, other methods (such as least squares) can also be used to compute the signal parameters from the original equations. Note however, that with most Gröbner basis algorithms, the above procedure is also followed (implicitly), as the signal parameters are eliminated first. The algorithm with explicit Gaussian elimination of the signal parameters is given in Algorithm 4. We will now illustrate this method for the setup used in Example 5.1.

Example 6.1 *(Fourier series using Gaussian elimination).* We use the same signal and sample values as in Example 5.1. Instead of calculating a Gröbner basis for the 28 subsets of 6 equations, we now eliminate the signal parameters first from the first $L = 5$ polynomials using Gaussian elimination. This gives us the signal parameters as a function of the offset:

$$\alpha_{-2} = 9.4824 + \frac{9.4824 - (0.0457 + 2.9289j)z_1 - 7.9982z_1^2 - (0.0457 - 2.9289j)z_1^3}{-1 + z_1^4},$$

$$\alpha_{-1} = -0.0457 - 2.9289j,$$

Algorithm 4. Algorithm for multichannel sampling from noisy samples using Gaussian elimination for the linear part.

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(1) Write out the equations from (5) describing the samples as a function of the signal coefficients.
(2) If necessary, perform a change of variables to convert the equations into a set of polynomial equations. These are linear in the signal coefficients $\boldsymbol{\alpha}$, and higher order polynomials in the offsets $\mathbf{t}$.
(3) Apply Gaussian elimination on the first L equations to compute the signal coefficients $\boldsymbol{\alpha}$ as a function of the offsets $\mathbf{t}$.
(4) Replace these values of $\boldsymbol{\alpha}$ in the remaining $MN - L$ equations and multiply each equation by its common denominator to obtain a set of $MN - L$ polynomial equations in the offsets $\mathbf{t}$.
(5) Divide these equations into at most $\binom{MN-L}{M-1}$ critical subsets of equations $\mathcal{S}_i$.
(6) Compute a Gröbner basis for each set $\mathcal{S}_i$.
(7) Calculate the possible offset values using back substitution and by computing the zeros of polynomial equations.
(8) Eliminate offset values that do not give a valid solution (e.g. values not on the unit circle in the Fourier case).
(9) Compute the weighted average of the offsets corresponding to the remaining solutions (typically one per set $\mathcal{S}_i$).
(10) Replace this value in the original equations and solve for the signal parameters $\boldsymbol{\alpha}$.

$$\begin{aligned}
\alpha_0 &= -5.90654,\\
\alpha_1 &= -0.0457 + 2.9289j,\\
\alpha_2 &= \frac{-9.48237 + (0.0457 + 2.9289j)z_1 + 7.9982z_1^2 + (0.0457 - 2.9289j)z_1^3}{-1 + z_1^4},
\end{aligned} \tag{50}$$

where we assume that $z_1^4 \neq 1$. We can then replace these values in the remaining three equations, and multiply them by their common denominators. This results in three polynomial equations in the unknown offset z_1:

$$\begin{aligned}
&(2.9746 + 2.8833j)z_1 + 6.4568z_1^2 + (2.9746 - 2.8833j)z_1^3\\
&\quad - (2.9746 + 2.8833j)z_1^5 - 6.4568z_1^6 - (2.9746 - 2.8833j)z_1^7 = 0,\\
&(-0.0913 - 5.8579j)z_1 - 1.3617z_1^2 - (0.0913 - 5.8579j)z_1^3\\
&\quad + (0.0913 + 5.8579j)z_1^5 + 1.3617z_1^6 + (0.0913 - 5.8579j)z_1^7 = 0,\\
&(-2.8833 + 2.9746j)z_1 - 5.4031z_1^2 - (2.8833 + 2.9746j)z_1^3\\
&\quad + (2.8833 - 2.9746j)z_1^5 + 5.4031z_1^6 + (2.8833 + 2.9746j)z_1^7 = 0.
\end{aligned} \tag{51}$$

As there is only a single unknown offset, the three possible critical subsets of equations that can be formed are the three separate equations. We do not need to compute a Gröbner basis for these subsets and can therefore directly compute the zeros for each of the polynomials separately. After elimination of the zeros that are not valid solutions (additional zeros were added by multiplying with the common denominators, the zeros have to be on the unit circle, etc.), we have the following zeros remaining for the three polynomials:

$$z_1^{(1)} = -0.9957 - 0.0924j, \qquad z_1^{(2)} = -0.9949 - 0.1007j, \qquad z_1^{(3)} = -0.9982 - 0.0594j. \tag{52}$$

From these values, we can compute the offsets $t_1^{(1)} = 2.0589$, $t_1^{(2)} = 2.0642$, and $t_1^{(3)} = 2.0378$. We take the average of these solutions ($t_{1,\mathrm{avg}} = 2.0537$), replace the corresponding value of z_1 in the original equations and compute a least squares solution to these linear equations. We obtain the coefficient vector

$$\hat{\boldsymbol{\alpha}} = \begin{pmatrix} 4.7412 - 5.7629j\\ -0.0457 - 2.9289j\\ -5.9065\\ -0.0457 + 2.9289j\\ 4.7412 + 5.7629j \end{pmatrix}. \tag{53}$$

The relative error for our estimation, $\|\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}\|/\|\boldsymbol{\alpha}\|$, is 0.655. This error can be compared to the error that would be obtained by applying a least squares estimation on the noisy samples with the exact offset t_1, which is 0.080. Averaged over 250 simulations with random coefficients and offsets, the estimated error norm is 0.618, compared to 0.095 with the exact offsets. We can see that our estimation has a larger error than in Example 5.1 (where the error was 0.340), but the computational complexity is also highly reduced. Instead of 28 Gröbner bases for sets of 8 equations, only 3 sets of a single equation remained, which could be directly solved.

Remark also that the computation of a Gröbner basis does not depend on the specific values of the samples, except in some degenerate cases. Once the size of the problem (L, M, N) is fixed, we could therefore compute the generic Gröbner basis for this setup. The first six steps from Algorithm 4 can then be precomputed. The online computations are reduced to steps 7–10: computing the zeros of a polynomial and replacing the solution(s) in the set of equations for the signal parameters. The zeros of a polynomial can be computed with the complexity given in (48). The other operations are negligible compared to this. Buchberger's algorithm is not needed anymore in the actual solution of the specific problem, which can be computed very efficiently.

7. Conclusions

In this paper, we have presented a method to reconstruct a signal from multiple sets of unregistered, aliased samples using Gröbner bases. First, we have shown how multichannel sampling with unknown offsets can be written as a set of polynomial equations. This was shown both for a polynomial signal and for a signal described by its Fourier series. Next, we applied Buchberger's algorithm to compute a Gröbner basis for the ideal corresponding to this set of equations. From a Gröbner basis, we can easily derive the unknown signal parameters. We presented an adaptation of our algorithm to the case of noisy measurements. Gröbner bases are then computed for critical subsets of polynomials. It is important to note however that this method is not very stable. Finally, some complexity issues were discussed, and a more efficient method was presented that computes the linear signal parameters first, such that a Gröbner basis has to be computed only for a much smaller set of equations in the unknown offsets. A Mathematica implementation of these methods is available online (http://lcav.epfl.ch/reproducible_research/SbaizVV06).

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