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Pricing and hedging in the presence of extraneous risks☆

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Abstract

Given an underlying complete financial market, we study contingent claims whose payoffs may depend on the occurrence of nonmarket events. We first investigate the almost-sure hedging of such claims. In particular, we obtain new representations of the hedging prices and provide necessary and sufficient conditions for a claim to be marketed. The analysis of various examples then leads us to investigate alternative pricing rules. We choose to embed the pricing problem into the agent's portfolio decision and study reservation prices. We establish the existence and consistency of this pricing rule in a semimartingale model. We characterize the nonlinear dependence of the reservation price with respect to both the agent's initial capital and the size of her position. The fair price arises as a limiting case.

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0. Introduction

We study the pricing and hedging of contingent claims whose payoffs are allowed to depend on the occurrence of nonmarket events. We start from a general semimartingale model of *complete markets* with respect to some market filtration $\mathbf{F} := (\mathcal{F}_t)_{t \leq T}$ and assume that there are extraneous risks, which we model by considering an observed filtration $\mathbf{G} := (\mathcal{G}_t)_{t \leq T}$ with $\mathbf{F} \subseteq \mathbf{G}$.

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In the presence of such risks, markets are incomplete and perfect hedging is, in general, no longer possible. Following [21,20] we show that the absence of arbitrage gives rise to a price interval whose endpoints correspond to the hedging prices of the claim. Extending previously known results, we establish new representations of these hedging prices and provide a set of necessary and sufficient conditions for claims to be attainable. In particular, we show that the upper hedging price need not be given by the supremum of the expectations over the whole set of martingale measures. We then apply these general results to specific examples, including a Brownian model with partial observation and a reduced-form model of event risk.

Since almost-sure hedging of the claim is only achievable at the endpoints of the arbitrage free interval, any price strictly contained in the arbitrage free interval leads to possible losses at the terminal time. As a result, the choice of a particular price can only be made with respect to a risk function representing the attitude towards risk and the endowment of the price setter. Relying on this observation, we investigate the *reservation prices*. Consider an agent who commits to sell a contingent claim and chooses his optimal portfolio so as to maximize his expected utility. We define the *reservation selling price* as the smallest amount which, when added to his initial wealth, allows him to achieve at least as high a level of expected utility as that he would have obtained without selling the contingent claim.[1]

Contrary to [8], who deal with the numerical computation of prices in a model with transaction costs, and to [17,22], who considered only the special case of the exponential utility function, we investigate theoretical properties of the reservation prices for Von-Neumann Morgenstern utility functions which are defined on the positive real line and satisfy the Inada conditions. After providing sufficient conditions for existence of the pricing rule, we show that the reservation prices define a closed interval which is contained in the arbitrage free interval. Further, we characterize the dependence of the reservation prices with respect to claims, the initial capital and the size of the position.

Central to our analysis is the *risk-neutral* price, which puts a zero risk-premium on all extraneous sources of risk. We show that this risk-neutral price, which depends neither on the agent's utility function nor on his initial wealth, coincides with the common limit of (i) the reservation prices as initial wealth increases and (ii) the reservation prices per unit of contingent claim as the size of the position decreases. Furthermore we show that this risk-neutral price is identical to the *fair price* which was introduced by [7] as the price a risk-averse individual would accept to pay for taking on an infinitesimal position in the security.

The remainder of the paper is organized as follows. Section 1 presents the framework. Section 2 clarifies the structure of the set of martingale measures. Section 3 studies the almost-sure hedging of contingent claims. Section 4 provides a counterexample supporting the results of Section 3, and discusses two examples of models with extraneous risks. Section 5 introduces the reservation prices and provides a detailed analysis of the associated utility maximization problems and the properties of the reservation prices.

1. The model

1.1. Traded securities

We consider a finite horizon model of a financial market which consists in $n + 1$ securities: one locally riskless savings account and n stocks.

[1] The reservation buying price is defined similarly. It is the maximal amount that the agent can accept to pay for the contingent claim while being sure to achieve at least as high a level of expected utility as that he would have obtained without doing so.

We assume that the price of the first asset is identically equal to one so that the capital invested in or borrowed from the savings account is constant over time. The price process

$$S := \{(S_t^i)_{i=1}^n : 0 \leq t \leq T\} \tag{1}$$

of the available stocks is assumed to be a nonnegative semimartingale on a probability space $(\Omega, \mathcal{G}, \mathbf{G}, P)$, where the filtration $\mathbf{G} := (\mathcal{G}_t)_{t \leq T}$ satisfies $\mathcal{G} = \mathcal{G}_T$ as well as the usual conditions of right continuity and completeness.

Remark 1. The assumption that the price of the savings account is constant simply means that we are working in discounted terms: all amounts have to be understood as being expressed in units of time-0 money. This normalization is harmless when studying arbitrage and almost-sure hedging occurs since these issues are essentially invariant under a change of numéraire, see [9] and [4, Chapter 10]. On the other hand, and as pointed out by the referee, this assumption has a quantitative impact on the utility based pricing of derivatives, unless one assumes that the agent's preferences are specified over discounted wealth. However, since our utility based pricing results are of a qualitative nature, this assumption does not hinder the generality of our findings.

Remark 2. The assumption that the stock price process is nonnegative is made for technical reasons. It allows us to work with general semimartingales while avoiding the subtleties associated with the difference between local martingales and σ-local martingales, see [11] for details. We could have obtained the same simplification by assuming that the process S is bounded from below but, since we like to think of the traded assets as limited liability stocks, it seems more natural to assume that their price is nonnegative.

1.2. Extraneous risks

Let $\mathbf{F} := (\mathcal{F}_t)_{t \leq T}$ be the usual augmentation of the filtration generated by the stock price process. With this definition, $\mathcal{G}_t$ represents *all* the information available to agents at time t, while $\mathcal{F}_t \subset \mathcal{G}_t$ represents the information which can be inferred from the observation of market prices only up to time t.

Remark 3. If, as is customary in finance models, the price process of the savings account was given by

$$S_t^0 = \exp\left[\int_0^t r_s \mathrm{d}s\right]$$

for some nonzero interest rate process r, then we would need to assume that the filtration $\mathbf{F}$ is generated by both S^0 and the discounted stock price S. The arbitrage and almost sure hedging results of Sections 2 and 3 remain valid in this slightly more general context, provided that Assumption 1 holds and that one imposes conditions on r which guarantee that the process S^0 is strictly positive and finite, see [4, Lemma 10.13 and Theorem 10.14].

In order to model extraneous risks whose occurrence can be influenced, but not completely determined, by market factors, we assume that

$$\mathcal{L} := \{A \subseteq \Omega : A \in \mathcal{G}_T \text{ but } A \notin \mathcal{F}_T\} \neq \emptyset.$$

Such events could, for example, be related to operational risks, the default of some agent, the occurrence of a storm or the prepayment of a prespecified number of loans in a pool of mortgage backed securities.

Remark 4. Unless $\mathcal{F}_t = \mathcal{G}_t$ for all $t \in [0, T]$, the case $\mathcal{L} = \emptyset$ models a situation where agents have some information about the future evolution of market prices, and hence corresponds to a model of insider trading. See [1] for an example of such of model.

Definition 1. Let $\mathbf{H} = (\mathcal{H}_t)_{t \leq T}$ denote either the market filtration $\mathbf{F}$ or the filtration $\mathbf{G}$. A probability measure Q is an $\mathbf{H}$-martingale measure if

(1) It is equivalent to the objective measure P on $\mathcal{G}$.
(2) Its Radon–Nikodym derivative with respect to the objective measure P on $\mathcal{G}$ is measurable with respect to $\mathcal{H}_T$.
(3) The process S is an $\mathbf{H}$-local martingale under Q.

Throughout the rest of the paper we denote by $\mathcal{M}_{\mathbf{H}}$ the set of $\mathbf{H}$-martingales measures and impose the following:

Assumption 1. The set $\mathcal{M}_{\mathbf{G}}$ is nonempty, and we have $\mathcal{M}_{\mathbf{F}} = \{Q\}$ for some probability measure Q.

As is well known, the first part of this assumption is essentially equivalent to the absence of arbitrage opportunities in the financial market, see [11] for a precise statement as well as for further references. On the other hand, the assumption that the set $\mathcal{M}_{\mathbf{F}}$ is a singleton implies that the financial market is complete with respect to its internal filtration, and allows to focus on the impact of extraneous risks.

Remark 5. Because $\mathbf{F}$-local martingales need not be $\mathbf{G}$-local martingales, the fact that $\mathcal{M}_{\mathbf{F}} = \{Q\}$ does not imply that the set $\mathcal{M}_{\mathbf{G}}$ is nonempty. In other words, the probability measure Q is an $\mathbf{F}$-martingale measure but it need not be a $\mathbf{G}$-martingale measure, see Section 2.

1.3. Trading strategies

A self-financing portfolio is defined as a pair (x, H) where $x \in \mathbf{R}$ represents the initial capital and $H = (H^i)_{i=1}^n$ is a $\mathbf{G}$-predictable and S-integrable process specifying the number of shares of each stock held in the portfolio.

The value process X of a self-financing portfolio evolves in time as the stochastic integral of the predictable process H with respect to the stock prices:

$$X_t := x + (H \cdot S)_t = x + \int_0^t H_\tau \mathrm{d}S_\tau. \tag{2}$$

As defined up to now, trading strategies are subject to very weak restrictions. In particular, the corresponding wealth process can be negative, and it is well known that in such a situation one must impose additional restrictions in order to exclude arbitrage opportunities from the market.

Definition 2. A process X with $X_0 = x \in \mathbf{R}_+$ is said to be admissible, and we write $X \in \mathcal{X}(x)$ if it is the value process of a self-financing trading strategy and is almost-surely nonnegative.

To handle the case of claims with possibly negative payoffs such as swaps and forward contracts, we need to consider a slightly larger set of trading strategies. Let us first recall from [10] that an admissible process X with initial value x is said to be *maximal* if its terminal value cannot be dominated by that of another admissible process, in the sense that $X' \in \mathcal{X}(x)$ and $X'_T \geq X_T$ imply $X_t = X'_t$ almost-surely for all $t \in [0, T]$.

Definition 3. A process X with $X_0 = x \in \mathbf{R}$ is said to be acceptable if it can be written as $X' - X''$ for some admissible process X' with initial value x' and some maximal admissible process X'' with initial value $x - x'$. In what follows, we denote by $\mathcal{A}(x)$ the set of acceptable processes with initial value $x \in \mathbf{R}$.

Acceptable processes were first introduced by [10] and have been used by [9] to solve some arbitrage problems in the context of financial market models with multiple currencies. Their relevance in the context of hedging and utility maximization problems will be demonstrated in later sections.

2. Martingale measures

As observed in Remark 5, the unique $\mathbf{F}$-martingale measure Q might not belong to the set of $\mathbf{G}$-martingale measures. Nevertheless, and as we now demonstrate, it plays a central role in the characterization of this set.

Theorem 1. *A probability measure R which is equivalent to P on $\mathcal{G}$ is a $\mathbf{G}$-martingale measure if and only if*

(1) *The restriction of R to $\mathcal{F}_T$ is equal to the restriction of Q to $\mathcal{F}_T$.*
(2) *Every $(\mathbf{F}, R)$-local martingale is a $(\mathbf{G}, R)$-local martingale.*

Proof. Assume for the "only if" part that $R \in \mathcal{M}_{\mathbf{G}}$, and denote by $Z(R)$ its density process with respect to Q on $\mathbf{G}$. To establish the first part, we need to prove that the $\mathbf{F}$-optional projection

$$\pi_t(R) := E_Q[Z_t(R)|\mathcal{F}_t] = E_Q\left[\left.\frac{\mathrm{d}R}{\mathrm{d}Q}\right|\mathcal{F}_t\right]$$

is identically equal to one. As is easily seen, this process is a $(\mathbf{F}, Q)$-martingale, and since Q is the unique probability measure with $\mathbf{F}$-measurable density under which the stock price process is an $\mathbf{F}$-local martingale, it follows from Jacod's representation theorem [13, p. 379] that

$$\pi_t(R) = \pi_0(R) + (\vartheta \cdot S)_t = 1 + (\vartheta \cdot S)_t$$

for some $\mathbf{F}$-predictable and S-integrable process ϑ. Using this in conjunction with Itô's lemma and the definition of $\mathcal{M}_{\mathbf{G}}$, we deduce that $\pi(R)Z(R)$ is a non negative $(\mathbf{G}, Q)$-local martingale and hence a supermartingale, and it follows that

$$1 = \pi_0(R)Z_0(R) \geq E_Q[\pi_t(R)Z_t(R)], \quad 0 \leq t \leq T. \tag{3}$$

On the other hand, the definition of the optional projection and the law of iterated expectations imply that

$$E_Q[\pi_t(R)Z_t(R)] = E_Q[\pi_t(R)E_Q[Z_t(R)|\mathcal{F}_t]] = E_Q[\pi_t^2(R)]$$

and combining this with Eq. (3) we obtain

$$E_Q[\pi_t^2(R)] \leq 1, \quad 0 \leq t \leq T.$$

Applying Jensen's inequality to the left hand side of the above expression, and using the fact that $\pi(R)$ is a martingale with initial value one, we conclude that the reverse inequality also holds, and it follows that $\pi_t(R) = 1$. To establish the second part, let M denote a $(\mathbf{F}, R)$-local martingale. Using the first part of the proof, we deduce that this process is also a ($\mathbf{F}$, Q)-local martingale, and it thus follows from Jacod's representation theorem that

$$M_t = M_0 + (\theta \cdot S)_t$$

for some $\mathbf{F}$-predictable integrand. Coming back to the definition of $\mathcal{M}_{\mathbf{G}}$, it is then easily seen that M is a ($\mathbf{G}$, R)-local martingale, and the desired result now follows from the arbitrariness of the local martingale M.

To establish the "if" part, we only have to show that the assumptions of the statement imply that the stock price is an ($\mathbf{F}$, R)-local martingale. To this end, let $(\tau_n)_{n=1}^{\infty}$ be a sequence of $\mathbf{F}$-stopping times which reduces the ($\mathbf{F}$, Q)-local martingale S. Using the first assumption, we obtain that

$$\begin{aligned} E_R[S_{\tau_n \wedge \tau}] &= E_Q[Z_{\tau_n \wedge \tau}(R) S_{\tau_n \wedge \tau}] \\ &= E_Q[\pi_{\tau_n \wedge \tau}(R) S_{\tau_n \wedge \tau}] = E_Q[S_{\tau_n \wedge \tau}] = S_0 \end{aligned}$$

for all $\mathbf{F}$-stopping time τ with values in $[0, T]$. By [13, Theorem 12, p. 83], this implies that for each $n \geq 1$, the stopped process $S_{\tau_n \wedge \cdot}$ is a ($\mathbf{F}$, R)-martingale, and it follows that the stock price is an ($\mathbf{F}$, R)-local martingale. □

The condition that $\mathbf{F}$-local martingales be $\mathbf{G}$-local martingales is known in filtering theory as hypothesis $\mathcal{H}$ and has been studied by [5]. In particular, these authors have shown that a sufficient condition for hypothesis $\mathcal{H}$ is that there exist a process which is both an $\mathbf{F}$ and a $\mathbf{G}$-local martingale, and which has the $\mathbf{F}$-martingale representation property. This simple result allows us to give necessary and sufficient conditions for the $\mathbf{F}$-martingale measure Q to be a $\mathbf{G}$-martingale measure.

Corollary 1. *The probability measure Q belongs to the set $\mathcal{M}_{\mathbf{G}}$ if and only if hypothesis $\mathcal{H}$ holds under Q, or equivalently under the objective measure.*

Proof. Assume that $Q \in \mathcal{M}_{\mathbf{G}}$. Applying Jacod's representation theorem, we have that all ($\mathbf{F}$, Q)-local martingales can be written as stochastic integrals with respect to the stock price, and observing that S is a ($\mathbf{G}$, Q)-local martingale, we conclude that hypothesis $\mathcal{H}$ holds under the probability measure Q. The converse follows directly from Theorem 1 and the fact that the Radon—Nykodim derivative of Q with respect to the objective measure is $\mathcal{F}_T$-measurable. □

3. Hedging prices

In this section we study the almost-sure hedging of general contingent claims, and provide necessary sufficient conditions for the attainability of a given claim. These results allow us to provide a static characterization of the set of terminal wealth that can be financed by an acceptable trading strategy, which we use extensively in Section 5.

Definition 4. A contingent claim is an element of the space $L^0 := L^0(\Omega, \mathcal{G}_T)$. Occasionally, we shall consider contingent claims whose payoffs are measurable with respect to $\mathcal{F}_T$. Such claims are referred to as $\mathbf{F}$-contingent claims.

Because the financial market is incomplete with respect to the filtration $\mathbf{G}$, the price of a contingent claim will not be uniquely defined in general, and we are thus naturally lead to consider almost-sure hedging.

Definition 5. The upper hedging price of the contingent claim B is

$$\hat{u}(B) := \inf\{x \in \mathbf{R} : \exists X \in \mathcal{A}(x) \text{ with } X_T \geq B\}. \tag{4}$$

Symmetrically, the lower hedging price of the contingent claim is the quantity defined by $\check{u}(B) := -\hat{u}(-B)$.

Remark 6. Using arguments similar to those of [20], it can be shown that the open interval $]\check{u}, \hat{u}[$ is arbitrage free in the sense that any price outside leads to an arbitrage opportunity while no price inside does.

The following well-known result gives a representation of the upper hedging price for a positive claim, and was originally proved by [2,9,21,23].

Proposition 1. *Let $D \in L^0_+$ and define $x := \sup_{R \in \mathcal{M}_\mathbf{G}} E_R[D]$. Then we have $\hat{u}(D) = x$ and if this nonnegative quantity is finite then there exists a maximal admissible process $X(D)$ in the set $\mathcal{X}(x)$ whose terminal value dominates D.*

Remark 7. Since we only assumed the stock prices to be local martingales under Q, it might be the case that $\hat{u}(S^k_T) < S^k_0$ for some k. As a result, one might be tempted to think that there exist arbitrage opportunities unless the stock prices are *real* martingales under Q.

To see that this is not the case, assume that $\hat{u}(S^k_T) < S^k_0$ for some k, and consider the self-financing portfolio obtained by selling one unit of the stock short and hedging the resulting exposure as prescribed by Proposition 1. The corresponding wealth process is given by

$$X_t = (S^k_0 - \hat{u}(S^k_T)) + X_t(S^k_T) - S^k_t$$

and has both a zero initial value and a nonnegative terminal value. Such a process thus constitutes a clear arbitrage provided that it is acceptable. But this can only be the case if the stock price is a martingale under Q, in which case we have $\hat{u}(S^k_T) = S^k_0$ and $X = 0$ everywhere.

The above result is only valid under the assumption that the claim is nonnegative (or bounded from below) and does not give any information on the lower hedging price unless the claim is bounded. The problem in dealing with general contingent claims is that we have to work with acceptable processes which, contrary to admissible processes, are not supermartingales under every martingale measure. To circumvent this difficulty, one needs to identify a subset of $\mathcal{M}_\mathbf{G}$ under which all the acceptable processes which dominate a given claim are supermartingales. This is the purpose of the following:

Lemma 1. *Let $D \in L^0_+$ be such that $\hat{u}(D)$ is finite, and let $X(D)$ be the maximal admissible process whose existence is asserted in Proposition 1. Then*

$$\mathcal{M}(D) := \{R \in \mathcal{M}_\mathbf{G} : X(D) \text{ is a } (\mathbf{G}, R)\text{-martingale}\}$$

is nonempty and convex. Furthermore, if X is an acceptable process such that $X_T + D$ is nonnegative, then X is a supermartingale under every $R \in \mathcal{M}(D)$.

Proof. See Appendix A. □

The next theorem generalizes most of the hedging results in the literature. It provides a representation of the hedging price as the supremum of expectations, not over the whole set $\mathcal{M}_\mathbf{G}$, but over a possibly strict subset (see Section 4.2 for a counterexample), and constitutes the main result of this section.

Theorem 2. *Let $B \in L^0$ be such that $-D \leq B$ for some nonnegative D whose upper hedging price is finite and define*

$$x := \sup_{R \in \mathcal{M}(D)} E_R[B]. \tag{5}$$

Then $\hat{u}(B) = x$, and if this quantity is finite then there exists a process in $\mathcal{A}(x)$ whose terminal value dominates B.

Proof. See Appendix A. □

In view of the above result, a natural question is that of knowing when we may take $\mathcal{M}(D) = \mathcal{M}_{\mathbf{G}}$. The following proposition gives a sufficient condition and establishes the completeness of the financial market with respect to $\mathbf{F}$.

Proposition 2. *Let D be a positive $\mathbf{F}$-contingent claim, as in Definition 4. Then we have $\hat{u}(D) = \check{u}(D) = E_Q[D]$ and $\mathcal{M} = \mathcal{M}(D)$.*

Proof. Consider the upper hedging price and assume $D \in L^1(Q)$, for otherwise there is nothing to prove. Using the first part of Theorem 1 in conjunction with Proposition 1, we have

$$E_R[D] = E_Q[D] = \hat{u}(D)$$

for all $R \in \mathcal{M}_{\mathbf{G}}$. Now let $X(D)$ denote the associated maximal admissible process. Using the first part of Theorem 1, and the fact that the process $X(D)$ is a supermartingale under all $\mathbf{G}$-martingale measures, we obtain that

$$E_R[D] \leq E_R[X_T(D)] \leq X_0(D) = E_Q[D] = E_R[D]$$

holds for all $R \in \mathcal{M}_{\mathbf{G}}$. This shows that the process $X(D)$ is a martingale under every $\mathbf{G}$-martingale measure and hence that $\mathcal{M}_{\mathbf{G}} = \mathcal{M}(D)$. By Remark 6 and the first part, we have $\check{u}(D) \leq E_Q[D]$. On the other hand, being a martingale under all martingale measures, the process $-X(D)$ is acceptable, and, since its terminal value is equal to $-D$, the result follows from Definition 5. □

Definition 6. A contingent claim B is said to be attainable or marketed if there exists a process X such that both X and $-X$ are acceptable, and $X_T = B$.

The next proposition gives necessary and sufficient conditions for a contingent claim to be attainable. It generalizes most of the attainability results in the literature, and constitutes the last result of this section.

Proposition 3. *Let B be a contingent claim such that $\hat{u}(|B|)$ is finite. Then B is attainable if and only if the mapping $R \mapsto E_R[B]$ is constant over $\mathcal{M}(|B|)$.*

Proof. See Appendix A. □

Remark 8. Apart from Proposition 2, all the results of this section are independent of the particular form of incompleteness studied in this paper. They are therefore valid for any semimartingale model of incomplete securities markets.

4. Examples

4.1. Partial observations

As a first example, we revisit the model of partial observations studied by [14,25] among others. In such a model, markets are complete with respect to the observed filtration, but, as we demonstrate below, the corresponding market martingale measure does not belong to the set $\mathcal{M}$.

Let $1 \leq n \leq d$, assume that the probability space supports a d-dimensional standard Brownian motion W, and denote by $\mathbf{G} := (\mathcal{G}_t)_{t \leq T}$ the usual augmentation of its natural filtration. The market consists of a savings account paying zero interest and n stocks whose prices satisfy

$$S_t = S_0 + \int_0^t \operatorname{diag}(S_u)(a_u \mathrm{d}u + \sigma \mathrm{d}W_u), \qquad S_0 \in (0, \infty)^n \tag{6}$$

for some $\mathbf{R}^n$-valued bounded appreciation rate process a, and some constant volatility matrix $\sigma \in \mathbf{R}^{n \times d}$ which is assumed to be of full rank.

Now consider an investor who does not observe the Brownian motions but only the stock prices, and let $\mathbf{F} := (\mathcal{F}_t)_{t \leq T}$ denote the usual augmentation of the filtration generated by his observations. Relying on well-known arguments from filtering theory, we have that

$$B_t := \Sigma^{-1} \left[\sigma W_t + \int_0^t (a_u - E[a_u | \mathcal{F}_u]) \mathrm{d}u \right], \qquad t \in [0, T]$$

is an n-dimensional $(\mathbf{F}, P)$-Brownian motion where $\Sigma \in \mathbf{R}^{n \times n}$ is the unique nonsingular matrix such that $\Sigma \Sigma^* = \sigma \sigma^*$. As a result, the price processes of the stocks can be written as

$$S_t = S_0 + \int_0^t \operatorname{diag}(S_u)(E[a_u | \mathcal{F}_u] \mathrm{d}u + \Sigma \mathrm{d}B_u). \tag{7}$$

The modified volatility matrix Σ being invertible, we then easily deduce that the equivalent probability measure defined by

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\mathcal{F}_t} = Z_t := \exp\left[-\int_0^t (\Sigma^{-1} E[a_u | \mathcal{F}_u])^* \left\{ \mathrm{d}B_u - \frac{1}{2} \Sigma^{-1} E[a_u | \mathcal{F}_u] \mathrm{d}u \right\} \right]$$

is the unique $\mathbf{F}$-martingale measure, and it follows that the market is complete when restricted to its internal filtration. In the full information case, the market is either complete or incomplete with respect to $\mathbf{G}$ depending on whether $n = d$ or not, but, as shown by the following proposition, the market martingale measure Q never belongs to the set $\mathcal{M}_{\mathbf{G}}$ unless $\mathbf{F} = \mathbf{G}$.

Proposition 4. *Assume that the process a is adapted to $\mathbf{G}$ but not to the market filtration $\mathbf{F}$. Then we have $Q \notin \mathcal{M}_{\mathbf{G}}$.*

Proof. Assume that the drift a is adapted to $\mathbf{G}$ but not to $\mathbf{F}$, and consider the uniformly integrable $(\mathbf{F}, P)$-martingale Z. Applying Itô's lemma and using the definition of the process B, we obtain:

$$\mathrm{d}Z_t = -Z_t E[a_t | \mathcal{F}_t]^* [\sigma \sigma^*]^{-1} [\sigma \mathrm{d}W_t + \{a_t - E[a_t | \mathcal{F}_t]\} \mathrm{d}t].$$

Since W has the martingale representation property with respect to $\mathbf{G}$, this implies that Z fails to be a $(\mathbf{G}, P)$-local martingale unless a is measurable with respect to the market filtration. As a result, hypothesis $\mathcal{H}$ fails to hold under the objective probability measure unless a is measurable with respect to the market filtration, and the rest now follows from Corollary 1. □

4.2. A counterexample

In this section we construct an attainable contingent claim whose lower hedging price is strictly larger than the infimum of its expectations over the set of martingale measures. This example, which follows directly from the results in [12], shows that the restriction to the set $\mathcal{M}(D)$ when computing the hedging prices of arbitrary claims cannot be avoided.

Let $\mathbf{G} := (\mathcal{G}_t)_{t\in\mathbf{R}_+}$ be the augmentation of the filtration generated by a two dimensional standard Brownian motion with coordinates $(Z^i)_{i=1}^2$, and let

$$S_t = \mathcal{E}_t(Z^1) = \exp\left(Z_t^1 - \frac{1}{2}t\right), \quad t \in \mathbf{R}_+$$

so that the market filtration coincides with the augmentation of the filtration generated by the first Brownian motion. As is easily seen, the stock price is both an $\mathbf{F}$ and a $\mathbf{G}$-local martingale under the objective measure, so hypothesis $\mathcal{H}$ holds under the objective measure and $P \in \mathcal{M}_{\mathbf{G}} \cap \mathcal{M}_{\mathbf{F}}$. Note, however, that because $\lim_{t\to\infty} S_t = 0$ almost surely, the stock price cannot be a uniformly integrable martingale under the objective probability measure.

Proposition 5. *There is a nonnegative random variable $B \in L^0(\Omega, \mathcal{G}_\infty)$, and an equivalent probability measure $R^* \in \mathcal{M}_{\mathbf{G}}$, with the property that*

$$\inf_{R\in\mathcal{M}_{\mathbf{G}}} E_R[B] \leq E[B] < 1 = E_{R^*}[B] = \check{u}(B) = \hat{u}(B). \tag{8}$$

Proof. Define an almost-surely finite $\mathbf{G}$-stopping time by setting

$$\vartheta := \inf\left\{t \in \mathbf{R}_+ : \mathcal{E}_t(Z^1) = \frac{1}{2} \text{ or } \mathcal{E}_t(Z^2) = 2\right\}$$

and let $B := S_\vartheta$. Since the stock price is a nonnegative supermartingale under any martingale measure, it follows from Proposition 1 that we have $\hat{u}(B) \leq 1$. On the other hand, it follows directly from [12, p.5] that the stopped process $\mathcal{E}^\vartheta(Z^2)$ is a uniformly integrable $(\mathbf{G}, P)$-martingale. We may thus define an equivalent probability measure by:

$$R^*(A) := E[1_A \mathcal{E}_\vartheta(Z^2)], \quad A \in \mathcal{G}_\infty.$$

Using [12, Theorem 2.1], we have that R^* is a martingale measure, and that the process $X_t := S_{t\wedge\vartheta} = \mathcal{E}_t^\vartheta(Z^1)$ is a uniformly integrable $(\mathbf{G}, R^*)$-martingale. This implies that the probability measure R^* belongs to the set $\mathcal{M}(B)$, and it now follows from the first part of the proof that

$$1 = X_0 = E_{R^*}[X_\infty] = \max_{R\in\mathcal{M}(B)} E_R[B] = \hat{u}(B) = \check{u}(B)$$

where the last equality is a consequence of Proposition 3. To complete the proof, it now only remains to check that $E[X_\infty] < 1$, but this follows once again from the results of [12]. □

4.3. Event risk

For our third example, we consider a model where the payoff of contingent claims depend on the occurrence of an extraneous event, which we model using a progressive enlargement of the market filtration.

Let $S := (S^i)_{i=1}^d$ be an arbitrary semimartingale satisfying the conditions of Assumption 1, and fix a strictly positive $\mathcal{G}$-measurable random variable τ. The observed filtration that we shall consider throughout this section is

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\{\tau \le r\} : 0 \le r \le t).$$

As is well known (see [6]), the filtration $\mathbf{G}$ is the smallest filtration which contains $\mathbf{F}$ and for which the random variable τ is a stopping time. Throughout this section, we shall impose the following:

Assumption 2. The stock price does not jump at time τ; that is, $\Delta S_\tau = 0^n$, and hypothesis $\mathcal{H}$ holds under the objective probability measure.

Define a $\mathbf{G}$-adapted increasing process by setting $N_t := 1_{\{\tau \le t\}}$. The first part of the above assumption guarantees that the stock price process and N are orthogonal, in the sense that $[S, N]$ is equal to zero. Now let A^0 denote the $(\mathbf{G}, Q)$-compensator of N, that is the unique $\mathbf{G}$-predictable, increasing finite variation process such that

$$M_t^0 := N_t - A_t^0 = 1_{\{\tau \le t\}} - A_t^0$$

is a $(\mathbf{G}, Q)$-local martingale. The following lemma characterizes a subset of the set of martingale measures.

Lemma 2. *Assume that the process A^0 is such that* $\exp A_T^0 \in L^1(Q)$. *Then, for each constant $\phi \in (-1, \infty)$, the nonnegative $(\mathbf{G}, Q)$-local martingale*

$$Z_t^\phi := \mathcal{E}_t(\phi M^0) = (1 + \phi N_t)\mathrm{e}^{-\phi A_t^0} \tag{9}$$

is a strictly positive, uniformly integrable $(\mathbf{G}, Q)$-martingale, and the formula $R^\phi(A) := E_Q[1_A Z_T^\phi]$ defines a $\mathbf{G}$-martingale measure.

Proof. The case $\phi = 0$ follows directly from the second part of Assumption 2 and Corollary 1. Now fix an arbitrary $\phi \in (-1, \infty) \setminus \{0\}$. In order to prove that Z^ϕ is a uniformly integrable martingale, it suffices to show that it is of class D, but this easily follows from the assumption in the statement, since

$$\sup_{t \le T} |Z_t^\phi| \le (1 + \phi^+) \exp[\phi^- A_T^0] \le (1 + \phi^+) \exp[A_T^0].$$

In order to establish that R^ϕ defines an equivalent martingale measure and thus complete the proof, we need to show that the product $Z^\phi S$ is a $(\mathbf{G}, Q)$-local martingale. Applying Itô's product rule, we have that

$$\begin{aligned} \mathrm{d}(Z_t^\phi S_t) &= Z_{t-}^\phi \mathrm{d}S_t + S_{t-} \mathrm{d}Z_t^\phi + \mathrm{d}[Z^\phi, S]_t \\ &= Z_{t-}^\phi \mathrm{d}S_t + S_{t-} \mathrm{d}Z_t^\phi + \phi Z_t^\phi (\Delta S_\tau 1_{\{\tau \le t\}} - \mathrm{d}[A^0, S]_t) \\ &= Z_{t-}^\phi \mathrm{d}S_t + S_{t-} \mathrm{d}Z_t^\phi - \phi Z_t^\phi \mathrm{d}[A^0, S]_t \end{aligned}$$

where the third equality follows from Assumption 2. The process A^0 being predictable and of finite variation on compacts, it follows from Yoeurp's lemma [13, VII.3.6] that $[A^0, S]$ is a $(\mathbf{G}, Q)$-local martingale, and the desired result now follows from the fact that Q is a $\mathbf{G}$-martingale measure. □

Let (A, C) denote a pair of nonnegative, $\mathcal{F}_T$-measurable random variables. The event-sensitive contingent claim associated with the pair (A, C) is the nonnegative $\mathcal{G}_T$-measurable random variable defined by

$$B := 1_{\{\tau > T\}} A + 1_{\{\tau \le T\}} C. \tag{10}$$

Such contingent claims specify payoffs that are measurable with respect to the market filtration **F**, but whose actual payment is conditional on the realization of the nonmarket event τ. This definition encompasses a variety of contingent claims, such as credit derivatives, equity linked life insurance policies, vulnerable derivatives and mortgage backed securities.

Proposition 6. *Assume that the conditions of Lemma 2 hold, and let B be given by (10) for some* **F***-contingent claims with* $0 \le C \le A \in L^1(Q)$. *Then*

$$E_Q[C] = \check{u}(B) \le \hat{u}(B) = E_Q[A]. \tag{11}$$

Symmetrically, if $0 \le A \le C \in L^1(Q)$, then the above equation remains valid if we interchange the role of the nonnegative random variables A and C.

Proof. Let B be as in the first part of the statement, and consider its lower hedging price. Since $C \le B$, we have $\check{u}(C) = E_Q[C] \le \check{u}(B)$, where the first equality follows from Proposition 2. On the other hand, the random variable A being measurable with respect to the market filtration, we have that $\mathcal{M}(A) = \mathcal{M}_{\mathbf{G}}$ by Proposition 2, and it follows from Theorem 2 that

$$\begin{aligned} \check{u}(B) &= \inf_{R \in \mathcal{M}_{\mathbf{G}}} E_R[B] = \inf_{R \in \mathcal{M}_{\mathbf{G}}} E_R[1_{\{\tau > T\}} A + 1_{\{\tau \le T\}} C] \\ &= E_Q[C] + \inf_{R \in \mathcal{M}} E_Q[(A - C) R(\tau > T | \mathcal{F}_T)]. \end{aligned} \tag{12}$$

Let now $\phi \in (-1, \infty) \backslash 0$, denote by R^ϕ the corresponding martingale measure, and recall from Theorem 1 that we have

$$1 = E_Q[Z_T^\phi | \mathcal{F}_T] = E_Q[(1 + \phi N_T) \mathrm{e}^{-\phi A_T^0} | \mathcal{F}_T].$$

Using the above identity in conjunction with the definition of the probability measure R^ϕ, we then get

$$\begin{aligned} R^\phi(\tau > T | \mathcal{F}_T) &= E_Q[1_{\{\tau > T\}} \mathrm{e}^{-\phi A_T^0} | \mathcal{F}_T] \\ &= \left(1 + \frac{1}{\phi}\right) E_Q[\mathrm{e}^{-\phi A_T^0} | \mathcal{F}_T] - \frac{1}{\phi}. \end{aligned}$$

Plugging this back into (12), and using the result of Lemma 2 in conjunction with the bounded convergence theorem, we then obtain

$$\begin{aligned} \check{u}(B) - E_Q[C] &\le \inf_{\phi > -1} E_Q\left[(A - C)\left\{\left(1 + \frac{1}{\phi}\right) E_Q[\mathrm{e}^{-\phi A_T^0} | \mathcal{F}_T] - \frac{1}{\phi}\right\}\right] \\ &\le \lim_{\phi \to \infty} E_Q\left[(A - C)\left\{\left(1 + \frac{1}{\phi}\right) E_Q[\mathrm{e}^{-\phi A_T^0} | \mathcal{F}_T] - \frac{1}{\phi}\right\}\right] = 0 \end{aligned}$$

which is the desired inequality. The upper hedging price and the second part of the statement are established by similar arguments, we omit the details. □

The above proposition points to the weaknesses of the almost-sure hedging criterion when dealing with event-sensitive contingent claims. Because the agent is not allowed to make a loss on the contingent claim, the cheapest hedge consists in abstracting from the event risk and replicating the maximal exposure. This suggests the need for alternative pricing rules which embed the pricing problem into a wider investment problem, where losses on the contingent claim part are allowed to happen as long as they can be compensated by the liquid part of the agent's portfolio. Such a pricing rule is studied in the next section.

5. Reservation prices

If the contingent claim is not attainable, then arbitrage arguments alone are not sufficient to determine a unique price, and the problem becomes that of selecting a point in the associated arbitrage free interval. Relying on Dybvig's [15] observation that in such a setting the pricing problem cannot be separated from the agent's global portfolio decisions, we now study the reservation prices of contingent claims whose payoffs are subject to extraneous risks.

In order to facilitate the analysis of this section, we let $\mathcal{B}$ denote the set of contingent claims B such that $|B| \leq D$ for some nonnegative D whose upper hedging price is finite, and impose the following

Assumption 3. The probability measure Q belongs to $\mathcal{M}_{\mathbf{G}}$.

Consider an agent endowed with an initial capital $x > 0$ and whose preferences over terminal wealth are represented by an expected utility functional $X \mapsto E[U(X)]$. The real valued function U is referred to as the agent's utility function, and it will be assumed to satisfy the following:

Assumption 4. The function $U : (0, \infty) \to \mathbf{R}$ is strictly concave, increasing, and continuously differentiable. Furthermore, its asymptotic elasticity

$$AE(U) := \limsup_{x\to\infty} \frac{xU'(x)}{U(x)}$$

is strictly smaller than one, and it satisfies the so-called Inada conditions in that we have $U'(0) = \infty$ and $U'(\infty) = 0$.

The agent's *primary* portfolio choice problem is to find a trading strategy whose terminal value maximizes his expected utility. The agent's primary value function is accordingly defined by

$$v_0(x) := \sup_{X\in\mathcal{A}(x)} E\left[U\left(X_T\right)\right], \quad x > 0. \tag{13}$$

Now suppose that before choosing a trading strategy, the agent uses the amount r to buy a contingent claim $B \in \mathcal{B}$. His initial liquid capital is then $x - r$, and he now faces the *secondary* problem

$$v(x - r, B) := \sup_{X\in\mathcal{A}(x-r)} E\left[U(X_T + B)\right], \quad x - r > \hat{u}(-B). \tag{14}$$

Given initial capital x, the agent will be ready to buy the claim at price r as long as this trade allows him to improve on his utility index, that is as long as $v_0(x) \leq v(x - r, B)$. This naturally leads us to the following

Definition 7. For an agent with initial capital $x > 0$ and utility function U, the reservation buying price of a European contingent claim is defined by

$$r_b(x, B) := \sup\{r \in \mathbf{R} : v_0(x) \leq v(x - r, B)\}. \tag{15}$$

Symmetrically, the reservation selling price of a European contingent claim is the quantity defined by $-r_s(x, B) := r_b(x, -B)$.

5.1. Existence of the optimal strategy

Relying on the results of Section 3, we have that the agent's secondary investment problem can be written in the static form

$$v(z, B) = \sup_{\xi \in \mathcal{C}(z,B)} E[U(\xi)]$$

where

$$\mathcal{C}(z, B) := \{\xi \in L^0_+ : E_R[\xi - B] \leq z \text{ for all } R \in \mathcal{M}(D)\}, \tag{16}$$

is the feasible set associated with the agent's endowment. We observe for later use that, whenever nonempty, the set $\mathcal{C}(z, B)$ is convex and closed with respect to the topology of almost-sure convergence.

Theorem 3. *Let $B \in \mathcal{B}$, and assume that the primary value function v_0 is finitely valued. Then the following hold:*

(1) *The agent's secondary value function $z \mapsto v(z, B)$ is finitely valued and strictly concave on $\mathcal{U} := \mathbf{R}\backslash(-\infty, \hat{u}(-B)]$.*
(2) *For each fixed initial capital $z \in \mathcal{U}$, there exists a unique acceptable process $\widehat{X}(z, B) \in \mathcal{A}(z)$ which attains the supremum in (14).*
(3) *If $B = 0$, then the unique solution to the agent's primary problem is adapted to the market filtration and given by*

$$\widehat{X}_t(x) := E_Q\left[I\left(y_x \frac{\mathrm{d}Q}{\mathrm{d}P}\right) \middle| \mathcal{F}_t \right] \tag{17}$$

where I denotes the continuous and strictly decreasing inverse of the agent's marginal utility, and $y_x \in (0, \infty)$ is chosen such that $\widehat{X}_0(x) = x$.

Proof. Consider first the case where $B = 0$, and observe that we only need to establish the fact that the optimal terminal wealth is measurable with respect to the market filtration, since the remaining assertions in the statement are consequences of Theorems 2.0 and 2.2 in [24]. This will follow once we show that the no-contingent claim problem can be written as

$$v_0(x) = \sup_{h \in \mathcal{C}(x,0)} E\left[U(h)\right] = \sup_{h \in \mathcal{D}} E\left[U(h)\right] := w_0(x) \tag{18}$$

where $\mathcal{D}$ denotes the set of $\mathbf{F}$-measurable random variables in $\mathcal{C}(x, 0)$. By our definition of the set of hedgeable claims, we have $v_0(x) \geq w_0(x)$. On the other hand, using the concavity of the utility function in conjunction with the law of iterated expectations and Jensen's inequality, we deduce that

$$v_0(x) \leq \sup_{h \in \mathcal{C}(x,0)} E[U(E[h|\mathcal{F}_T])] = \sup_{g \in \mathcal{E}} E[U(g)]$$

where $\mathcal{E}$ is the projection of the set $\mathcal{C}(x, 0)$ on the terminal market filtration. As is easily seen from the definitions, we have $\mathcal{D} \subset \mathcal{E}$. In order to establish the reverse inclusion, and thus complete our proof, let $g = E[h|\mathcal{F}_T]$ for some nonnegative random variable $h \in \mathcal{C}(x, 0)$. Using the first part of Theorem 1 in conjunction with the fact that by construction $\mathcal{M}(0) = \mathcal{M}_{\mathbf{G}}$, we obtain that

$$E_R[g] = E_Q[g] = E\left[g\frac{\mathrm{d}Q}{\mathrm{d}P}\right] = E\left[h\frac{\mathrm{d}Q}{\mathrm{d}P}\right] = E_Q[h] \leq x$$

holds for all martingale measures. It follows that we have $g \in \mathcal{C}(x, 0)$ and hence $\mathcal{E} \subset \mathcal{D}$ since the nonnegative random variable g is measurable with respect to the terminal market filtration.

Under our assumptions, the results for the case $B \neq 0$ can be deduced from Theorem 2 and Corollary 1 in [18]. While these authors rely on duality theory to prove the existence and uniqueness of the optimal strategy, we provide in the appendix an alternative, direct argument for the reader's convenience. □

Remark 9. The assumption that $Q \in \mathcal{M}_{\mathbf{G}}$ cannot be dropped without losing the $\mathbf{F}$-measurability of the optimal primary trading strategy. To see this, consider the partial observations model with $n = d = 1$. In this model, the market is complete with respect to both filtrations, but the densities of the corresponding equivalent martingale measures differ as long as $\mathbf{F} \neq \mathbf{G}$.

5.2. Properties of the pricing rule

5.2.1. Consistency

We now check that the pricing rule derived from the definition of the reservation prices is consistent in the sense that (i) neither the reservation buying price nor the reservation selling price induce an arbitrage opportunity; and (ii) the reservation buying price is always smaller than the reservation selling price.

Proposition 7. *Let B denote an element of $\mathcal{B}$. Then its reservation prices exist, and the consistency condition*

$$\check{u}(B) \leq r_b(x, B) \leq r_s(x, B) \leq \hat{u}(B) \tag{19}$$

holds for any strictly positive initial capital.

Proof. By Theorem 3, the secondary value functions $v(\cdot, \pm B)$ are strictly concave and hence continuous on their respective domains. The existence of the reservation prices is therefore trivial, and furthermore we have that

$$v_0(x) = v(x - r_b(x, B), B) = v(x + r_s(x, B), -B) \tag{20}$$

holds for every strictly positive initial capital. Let us now turn to the second part, fix an initial capital, and consider the inequality $\check{u} := \check{u}(B) \leq r_b(x, B)$. The lower hedging price $\check{u}$ being finite, we know from Theorem 2 that there exists an acceptable process $X(-B)$ with initial value $-\check{u}$ whose terminal value dominates $-B$. On the other hand, using the fact that the set of acceptable processes is a cone, we obtain

$$v_0(x) \leq \sup_{X \in \mathcal{A}(x)} E[U(X_T + X_T(-B) + B)] \leq v(x - \check{u}, B)$$

and it follows that $\check{u} \leq r_b(x, B)$ holds. The last inequality in the statement follows from the symmetry in the definition of the reservation prices so all there remains to prove is that the reservation prices are consistent with one another. To this end, fix an initial capital $x \in (0, \infty)$ and let X^b (resp. X^s) denote the optimal acceptable process when the contingent claim is bought (resp. sold) at the reservation price. As is easily seen, the process $\overline{X} = \frac{1}{2}\left(X^b + X^s\right)$ is acceptable for the initial capital $\overline{x} := x + (r_s - r_b)/2$ and its terminal value is nonnegative. Using the result of Lemma 3, it is then easily deduced that this process is in fact admissible, and it now follows from the concavity of the agent's utility function that we have

$$\frac{1}{2} \cdot E[U(X_T^b + B) + U(X_T^s - B)] \leq E[U(\overline{X}_T)] \leq v_0(\overline{x}).$$

Using the definition of X^b and X^s in conjunction (20), it is easily deduced that left hand side of the above expression equals $v_0(x)$, and the desired result follows from the increase of the agent's utility function. □

As an immediate consequence of Propositions 2 and 7, we obtain that for an **F**-contingent claim D, the reservation prices depend neither on the agent's utility function nor on his initial capital, and that they coincide with the arbitrage free price $E_Q[D]$. More generally, we have the following:

Corollary 2. *Let (B_1, B_2) be contingent claims with the properties that B_2 is attainable, and the random variable $D := \max_i |B_i|$ belongs to $\mathcal{B}$. Then we have*

$$r_k(x, B) = r_k(x, B_1) + E_R[B_2], \quad k \in \{s, b\},\ R \in \mathcal{M}(D)$$

for any strictly positive initial capital. In particular, the reservation prices of the attainable claim B_2 are equal and given by $E_R[B_2]$ for all $R \in \mathcal{M}(D)$.

Proof. Let (B_i) be as in the statement, and fix a strictly positive initial capital x. The results of Proposition 3 show that $\check{u} = \hat{u} = E_R[B_2]$. Using the definition of attainable contingent claims in conjunction with the fact that workable processes are acceptable (both long and short), we deduce that

$$v_0(x) = v(x - E_R[B_2] - r_b(x, B_1), B)$$

holds for all $R \in \mathcal{M}(D)$, and the desired property of the reservation buying price now follows from Eq. (20) and the continuity of the secondary value function. The corresponding property of the selling price follows from the symmetry in the definition of the reservation prices. □

5.2.2. Comparative statics

Having obtained fairly general conditions guaranteeing the existence and consistency of the reservation prices for a wide class of contingent claims, we now turn to the study of some of their properties.

Proposition 8. *The following assertions hold:*

(1) *For a fixed $x \in (0, \infty)$, the reservation buying price $r_b(x, \cdot) : \mathcal{B} \to \mathbf{R}$ is increasing and concave with respect to contingent claims.*
(2) *For an arbitrary $B \in \mathcal{B}$ and an arbitrary $x \in (0, \infty)$, the reservation unit buying price is defined by*

$$r_{b1}(x, \delta, B) := \delta^{-1} r_b(x, \delta B), \quad \delta \in \mathbf{R}_+ \tag{21}$$

and is increasing with respect to the size of the agent's position.
(3) *For an arbitrary $B \in \mathcal{B}$, the reservation buying price converges to the lower hedging price as the initial capital decreases to zero.*

(4) *Assume that the agent has constant relative risk aversion and let $B \in \mathcal{B}$; then the reservation buying price $r_b(\cdot, B) : (0, \infty) \to \mathbf{R}$ is increasing with respect to the agent's initial capital.*

Remark 10. (a) Because of the symmetry in the definition of the reservation prices, each of the above properties of the buying price can be turned into a property of the selling price. In particular, it follows from the above results that, for a fixed initial capital, the reservation selling price is increasing and convex with respect to contingent claims.

(b) Using Corollary 2 and the fact that constants are attainable, we have that reservation prices are invariant under cash translations, and it follows that for any fixed $x \in (0, \infty)$ the mapping $\rho^x(B) := -r_b(x, B)$ defines a convex measure of risk in the sense of [16]. If the interest rate was nonzero, then the reservation prices would be invariant under "forward" cash translations of the form αS_T^0, where α is a constant and S_T^0 is the terminal value of the savings account (the reference instrument in the terminology of [3]).

Proof. Let $(B_i)_{i=1}^2 \in \mathcal{B}$ be contingent claims such that $B_1 \leq B_2$. As is easily seen from the definition of the sets $\mathcal{C}(\cdot|B_i)$, we have that the inclusion $\mathcal{C}(\cdot, B_1) \subset \mathcal{C}(\cdot, B_2)$ holds, and it thus follows from the increase of the utility function that

$$v(z, B_1) \leq v(z, B_2), \quad \text{for all } z > \hat{u}(-B_1).$$

In particular, taking z to be of the form $x - r_b(x, B_1)$ for some strictly positive initial capital and using (20) in conjunction with the above inequality, we obtain that $v_0(x) \leq v(z, B_2)$ holds, and it follows that we have $r_b(\cdot, B_1) \leq r_b(\cdot, B_2)$. To check for concavity of the reservation buying price, fix an arbitrary strictly positive initial capital x, and let $(B_i)_{i=1}^2 \in \mathcal{B}$. Further, let $z_i = x - r_b(x, B_i)$, and observe that from the definition of the feasible sets, we have

$$\lambda\mathcal{C}(z_1, B_1) + (1-\lambda)\mathcal{C}(z_2, B_2) \subseteq \mathcal{C}(\lambda z_1 + (1-\lambda)z_2, \lambda B_1 + (1-\lambda)B_2)$$

for all $\lambda \in [0, 1]$. Using this fact in conjunction with the result of Theorem 3 and the concavity of the agent's utility function and (20), we obtain that

$$v_0(x) \leq v(\lambda z_1 + (1-\lambda)z_2, \lambda B_1 + (1-\lambda)B_2)$$

holds for all $\lambda \in [0, 1]$, and the desired property of the reservation buying price now follows from the definition of the constants $(z_i)_{i=1}^2$ and the increase of the agent's utility function. The second assertion is an easy consequence of the first. Indeed, fix an arbitrary pair (x, B) and observe that we have

$$r_{b1}(x, \delta, B) = \delta^{-1} r_b(x, \delta B) \leq (\delta\lambda)^{-1} r_b(x, \delta\lambda B) = r_{b1}(x, \delta\lambda, B)$$

for all $(\lambda, \delta) \in [0, 1] \times \mathbf{R}_+$ by the concavity of the reservation buying price with respect to contingent claims. The third assertion being an easy consequence of Definition 7 and Theorem 2, we now turn to the fourth one. If the agent has constant relative risk aversion, then her utility function is given by either

$$U(x) = \alpha + \beta \log(x) \quad \text{or} \quad U(x) = \alpha + \beta(x^{1-\gamma}/(1-\gamma)) \tag{22}$$

for some $(\alpha, \beta, \gamma) \in \mathbf{R}_+^3$, $\gamma \neq 1$. Observing that $\mathcal{C}(\lambda z, \lambda B) = \lambda\mathcal{C}(z, B)$ holds for all nonnegative constants, and using standard properties of the constant relative risk aversion utility functions defined by (22), we get that

$$r_b(\lambda x, \lambda B) = \sup\left\{p \in \mathbf{R} : v_0(x) \leq v\left(x - \frac{p}{\lambda}, B\right)\right\}$$

holds for all strictly positive constants λ, and can thus conclude that for constant relative risk aversion functions the reservation buying price is positively homogeneous with respect to (x, B), in the sense that

$$r_b(\lambda x, \lambda B) = \lambda r_b(x, B), \quad \lambda \in (0, \infty). \tag{23}$$

In particular, taking $\lambda \geq 1$ and using (23) in conjunction with the concavity of the reservation buying price with respect to contingent claims, we obtain that the inequality $r_b(x, B) \leq r_b(\lambda x, B)$ holds, and our proof is complete. □

5.2.3. *Relations with the fair price*

Considering an incomplete model of financial markets, [7] defines the *fair price* (also referred to as marginal utility based price) by postulating a zero marginal rate of substitution for small positions. Instead of this differential concept, we use an optimality criterion, which will allow us to bypass the possible nonsmoothness of the agent's value function.

Definition 8. Let $B \in \mathcal{B}$, and fix an arbitrary initial capital $x \in (0, \infty)$. Then $p \in \mathbf{R}$ is said to be a fair price for the contingent claim given the agent's initial capital if the inequality $v(x - qp, qB) \leq v_0(x)$ holds for all $q \in \mathbf{R}$.

Remark 11. If the function $(z, q) \mapsto v(z, qB)$ was continuously differentiable on its effective domain, then the fair price of the contingent claim would simply be given by the unique solution to the equation

$$\left.\frac{\partial v}{\partial q}(x - qp, qB)\right|_{q=0} = 0 \tag{24}$$

and our definition of the fair price would coincide with the original definition in [7]. Unfortunately, this value function cannot be shown to be differentiable in general (see [19]), and this is precisely why we define the fair price through an optimality criterion. An alternative definition was given and studied by [20].

Roughly speaking, the fair price is defined in such a way that the agent is locally risk neutral at the optimum of the no-contingent claim problem. As a result, we expect it to be related to the agent's marginal utility at the optimum of the latter problem, that is to the market's martingale measure Q. The following result makes this intuition precise and sheds some light on the relations between the fair price and the reservation prices of a given claim.

Theorem 4. *Let $B \in \mathcal{B}$ be such that the probability measure $Q \in \mathcal{M}(D)$ for some nonnegative random variable $D \geq |B|$. Then we have*

(1) *The contingent claim admits a unique fair price, which is independent of (x, U) and given by $\widehat{p} := E_Q[B]$.*
(2) *For every strictly positive initial capital, the fair price of the contingent claim satisfies the inequalities $r_b(x, B) \leq \widehat{p} \leq r_s(x, B)$.*
(3) *For every strictly positive initial capital, the fair price of the contingent claim satisfies $\widehat{p} = \lim_{\delta \to 0} r_{b1}(x, \delta, B) = \lim_{\delta \to 0} r_{s1}(x, \delta, B)$.*
(4) *If the agent has constant relative risk aversion, then the fair price of the contingent claim satisfies $\widehat{p} = \lim_{x \to \infty} r_b(x, B) = \lim_{x \to \infty} r_s(x, B)$.*

Proof. In order to show that $\widehat{p}$ indeed defines a fair price, fix a constant q, and let $X \in \mathcal{A}(x - q\widehat{p})$ denote an acceptable process such that $X_T + qB \geq 0$. Using the concavity of the agent's utility function in conjunction with the third assertion of Theorem 3, we obtain that

$$\begin{aligned} E\left[U(X_T + qB)\right] &\leq E[U(\widehat{X}_T(x)) + U'(\widehat{X}_T(x))(X_T + qB - \widehat{X}_T(x))] \\ &= v_0(x) + y_x(E_Q[X_T + qB] - x) \leq v_0(x) \end{aligned}$$

where the last inequality follows from Lemma 3 and the fact that $Q \in \mathcal{M}(D)$ by assumption. Taking the supremum over acceptable processes on both sides, we obtain that $v(x - q\widehat{p}, qB) \leq v_0(x)$, and conclude that $\widehat{p}$ defines a fair price.

To prove the uniqueness of this fair price, we need to show that for any $p \neq \widehat{p}$, there exists $q \neq 0$ such that $v(x - qp, qB) > v_0(x)$. As is easily seen, the existence of such a quantity will follow once we have shown that

$$\partial := \liminf_{|q| \to 0} \left\{ \frac{-v_0(x) + v(x - qp, qB)}{|q|} \right\} > 0 \tag{25}$$

whenever $p \neq \widehat{p}$. Before doing so, let us first recall from Proposition 1 that there exists a maximal admissible process X' with initial value $x' = \hat{u}(D)$ whose terminal value dominates the random variable $|B|$, and that this process is a uniformly integrable martingale under Q by assumption. Coming back to the proof of (25), assume that we are given $p < \widehat{p}$. Let $(\varepsilon_n)_{n=1}^{\infty}$ with $1 > \varepsilon_n \to 0$, set $q_n := x\varepsilon_n/(x' + p + \varepsilon_n)$, and define a sequence of processes by setting

$$X^n := q_n X' + \left(\frac{\varepsilon_n + (1 - \varepsilon_n)(p + x')}{x' + p + \varepsilon_n} \right) \widehat{X}(x), \quad n \geq 1$$

where $\widehat{X}(x)$ is the solution to the no-contingent claim problem. Observing that for the given p to be arbitrage free we must have $p \in [-x', x']$, we immediately deduce that the quantity q_n is strictly positive, and it now follows from the definition of the set $\mathcal{B}$ that we have $X_T^n \in \mathcal{C}(x - q_n p | q_n B)$. Using this sequence in conjunction with the concavity of the utility function, we obtain

$$\begin{aligned} \partial &\geq \liminf_{n \to \infty} E\left[\left(X_T' + B - \frac{p + x'}{x} \widehat{X}_T(x) \right) U'(X_T^n + q_n B) \right] \\ &\geq E\left[\left(X_T' + B - \frac{p + x'}{x} \widehat{X}_T(x) \right)^+ U'(\widehat{X}_T(x)) \right] \\ &\quad - \limsup_{n \to \infty} E\left[\left(X_T' + B - \frac{p + x'}{x} \widehat{X}_T(x) \right)^- U'(X_T^n + q_n B) \right] \\ &\geq E\left[\left(X_T' + B - \frac{p + x'}{x} \widehat{X}_T(x) \right) U'(\widehat{X}_T(x)) \right] = y_x\,(\widehat{p} - p) \end{aligned} \tag{26}$$

where the second inequality follows from Fatou's lemma; the third inequality follows from the monotone convergence theorem; and the last equality follows from Theorem 3, the definition of $\widehat{p}$ and the fact that X' is a uniformly integrable martingale under Q. The right hand side being strictly positive by assumption, (25) holds, and it follows that p cannot be a fair price for the claim. The case where $\widehat{p} < p$ is treated similarly, so we omit the details.

The second assertion being an easy consequence of the first and Theorem 2, we omit its proof and turn to the third one. Let $\ell : (0, \infty) \to \mathbf{R}$ denote the reservation unit buying price as a function of the position's size, and observe that since ℓ is a decreasing and bounded function by Proposition 8 and Assertion (2), the limit $\ell(0)$ exists and is finite. Using the definition of the reservation prices in conjunction with an argument similar to that which lead to (26), we obtain

$$0 \geq \lim_{\delta \to 0} y_x (\widehat{p} - \ell(\delta)) = y_x (\widehat{p} - \ell(0))$$

and the desired equality follows, since we have from Assertion (2) that $\ell(\delta) \leq \widehat{p}$ holds for all strictly positive δ. The counterpart of this result for the reservation selling prices follows from Remark 10.

In order to complete the proof, assume that the agent has constant relative risk aversion, and recall from the proof of Proposition 8 that this implies

$$r_b(\lambda x, \lambda B) = \lambda r_b(x, B), \quad \lambda \in (0, \infty).$$

Using this in conjunction with the definition of the reservation unit buying price, we obtain that $r_b(x/\delta, B) = r_{b1}(x, \delta, B)$ holds for all strictly positive δ. In particular, taking the limit of the above expression as $x \to \infty$ we have

$$\lim_{x \to \infty} r_b(x, B) = \lim_{\delta \to 0} r_b(1/\delta, B) = \lim_{\delta \to 0} r_{b1}(1, \delta, B) = \widehat{p}$$

where the last equality is a consequence of Assertion (3). The counterpart for the selling prices again follows from Remark 10. □

Remark 12. (a) The assumption that $Q \in \mathcal{M}(D)$ cannot be relaxed without losing the uniqueness of the fair price. In fact, Hugonnier et al. [19] show that if this condition fails to hold, then there exists a contingent claim with $|B| \leq D$ and constants such that any $p \in [\alpha, \beta]$ constitutes a fair price for the contingent claim.

(b) If the claim under consideration is attainable, then no assumption is needed to obtain the result that its fair price coincides with its arbitrage free price, since in this case $v(z, qB) = v_0(z + qE_R[B])$ for all $R \in \mathcal{M}(D)$.

The above results imply that two agents having no previous position in the contingent claim can only agree on a limit price as their respective initial capitals increase to infinity or as the size of the deal becomes negligible. However, if one of the agents already has a position in some contingent claims, or is not able to trade freely in the market, then an equilibrium price quantity pair may exist. We plan to investigate such a framework in future research.

Appendix A. Proof of Lemma 1, Theorem 2 and Proposition 3

Before proceeding with the proofs of Lemma 1, Theorem 2 and Proposition 3 we start by establishing a characterization of the set of acceptable processes.

Lemma 3. *Let X be given by (2) for some portfolio (x, H). Then $X \in \mathcal{A}(x)$ if and only if the upper hedging price of X_T^- is finite and there exists at least one $R \in \mathcal{M}_{\mathbf{G}}$ under which X is a supermartingale.*

Proof. We start with the implication (ii) $\Rightarrow$ (i). Assume that X is a supermartingale under some martingale measure $R \in \mathcal{M}_{\mathbf{G}}$ and let

$$x'' := \hat{u}(X_T^-) = \sup_{U \in \mathcal{M}} E_U[X_T^-] < \infty \tag{27}$$

denote the initial value of the maximal admissible process X'' whose existence is asserted in the second part of Proposition 1. Using the fact that admissible processes are super martingales under all $R \in \mathcal{M}_{\mathbf{G}}$, we obtain

$$0 \leq E_R[X_T^+|\mathcal{G}_t] \leq E_R[X_T + X_T''|\mathcal{G}_t] \leq X_t + X_t'' =: X_t',$$

where the second inequality follows from the definition of X''. This shows that the process X' is admissible and writing X as the difference $X = X' - X''$ we conclude that Assertion (i) holds true.

Conversely, let X be acceptable and denote by X'' the corresponding maximal process. Observing that $X^- \leq X''$ and using the fact that the process X'' is admissible we easily deduce that (27) holds true. On the other hand, using the first part of Lemma 1 we have that there exists a probability measure $R \in \mathcal{M}_{\mathbf{G}}$ under which the process X'' is a martingale and the desired property now follows from the fact that admissible processes are supermartingales under all martingale measures. □

Proof of Lemma 1. The first part follows directly from [9, Theorem 13], we omit the details. For the second part, let X be as in the statement and consider the process $X' = X + X(D)$. By construction, we have

$$X_T' := X_T + X_T(D) \geq 0.$$

Using Lemma 3 in conjunction with the fact that $X(D)$ is a supermartingale under all martingale measures we then obtain that the process X' is almost-surely non negative and hence admissible. Finally, writing

$$X = X' - X(D) = (X + X(D)) - X(D)$$

and using the fist part in conjunction with the fact that admissible processes are supermartingales under all $R \in \mathcal{M}_{\mathbf{G}}$ we obtain that X is a supermartingale under all $R \in \mathcal{M}(D)$. □

Proof of Theorem 2. Let $x_0 \in \mathbf{R}$ be such that there exists an acceptable process $X \in \mathcal{A}(x_0)$ whose terminal value dominates the random variable B. Using the fact second part of Lemma 1 we obtain

$$E_R[B] \leq E_R[X_T] \leq x_0, \quad R \in \mathcal{M}(D).$$

Taking successively the supremum over $R \in \mathcal{M}(D)$ on the left hand side and the infimum over x_0 on the right hand side, we conclude that $x \leq \hat{u}(B)$. In order to establish the reverse inequality, we assume that x is finite for otherwise there is nothing to prove. Let $\widetilde{X}$ with

$$\begin{aligned} \widetilde{X}_0 &:= \sup_{R \in \mathcal{M}_{\mathbf{G}}} E_R[B + X_T(D)] = \sup_{R \in \mathcal{M}(D)} E_R[B + X_T(D)] \\ &= \sup_{R \in \mathcal{M}(D)} E_R[B] + X_0(D) = x + X_0(D) \end{aligned} \tag{28}$$

denote the maximal admissible process associated with the positive contingent claim $B + X_T(D)$ and define an acceptable process by setting $X := \widetilde{X} - X(D)$. As is easily seen, we have $X_T \geq B$ and since the initial value of this process is equal to x we conclude that the desired inequality holds. □

Proof of Proposition 3. To establish the implication (1) $\Rightarrow$ (2), assume that B is attainable and denote the corresponding workable process by X. Applying Lemma 3 to $+X$ and $-X$, we deduce that X is martingale under all $R \in \mathcal{M}(D)$ and this implies the validity of the second assertion. In order to establish the reverse implication assume that

$$x := \inf_{R \in \mathcal{M}(D)} E_R[B] = \sup_{R \in \mathcal{M}(D)} E_R[B]$$

and denote by $X \in \mathcal{A}(x)$ with $X_T \geq B$ the corresponding super-replicating process. Using the fact that this process is a supermartingale under every probability measure $R \in \mathcal{M}(D)$ we obtain

$$x \geq E_R[X_T] \geq E_R[B] = x$$

and conclude that X is a uniformly integrable R-martingale with terminal value equal to B. Combining this with the result of Lemma 3 we easily deduce that the processes $\pm X$ are acceptable and our proof is complete. □

Appendix B. Proof of Theorem 3

Assume $\check{u}(B) < \hat{u}(B)$ for otherwise the result follows from the first part and Proposition 3. As is easily seen from the definition of the set $\mathcal{C}(z, B)$ and that of the hedging prices, we have

$$v_0(z + \check{u}(B)) \leq v(z, B) \leq v_0(z + \hat{u}(B)), \quad z \in \mathcal{U} \tag{29}$$

and it follows that the agent's secondary value function is finite. Furthermore, this function being increasing and concave, there is a finite positive limit

$$\ell := \lim_{c \to \infty} \frac{v(z + c, B)}{c}.$$

Using the right hand inequality in (29) in conjunction with de l'Hospital rule and the first part we obtain

$$\ell \leq \lim_{c \to \infty} \frac{v_0(z + c + \hat{u}(B))}{c} = \lim_{x \to \infty} \frac{v_0(x)}{x} = 0$$

where the last equality follows from Theorem 2.2 in [24] and conclude that $\ell = 0$. Now let $(h^n)_{n \geq 1}$ be a maximizing sequence for the agent's problem with initial capital $z \in \mathcal{U}$. Using the fact that the set $\mathcal{C}(z, B)$ is convex and closed with respect to almost-sure convergence in conjunction with Komlòs lemma, we deduce that there is a subsequence $(h^k)_{k \geq 1}$ of convex combinations which converges almost-surely to some positive random variable $h^* \in \mathcal{C}(z, B)$.

We claim that this random variable is in fact optimal. From the concavity of the agent's utility function we deduce that the subsequence $(h^k)_{k \geq 1}$ is still maximizing. On the other hand, applying Fatou's lemma we obtain

$$E[U(h^*)^-] \leq \liminf_{k \to \infty} E[U(h^k)^-]$$

and conclude that the optimality of h^* will follow once we have shown that the sequence $(U(h^k)^+)_{k \geq 1}$ is uniformly integrable. To show that this is indeed the case, we assume that $U(\infty) > 0$ for otherwise there is nothing to prove and suppose to the contrary that $(U(h^k)^+)_{k \geq 1}$ is not uniformly integrable. Then, passing to a subsequence if necessary, we can find a strictly positive α and a sequence $(A_k)_{k \geq 1}$ of pairwise disjoint subsets of the probability space such that

$$\alpha \leq E[1_{A_k} U(h^k)^+], \quad k \geq 1. \tag{30}$$

Let x_0 denote the smallest strictly positive x such that $U(x) \geq 0$ and define a sequence $(c^n)_{n\geq 1}$ of non negative random variables by setting:

$$c^n := x_0 + \sum_{k=1}^{n} 1_{A_k} h^k, \quad n \geq 1.$$

Recall that the claim satisfies $|B| \leq D$ for some nonnonnegative random variable whose upper hedging price is finite and let R denote an arbitrary equivalent martingale measure in $\mathcal{M}(D)$. Using the definition of the set $\mathcal{C}(z, B)$ we obtain

$$E_R\left[c^n - B\right] \leq x_0 + n(z + \hat{u}(D)) := z^n$$

and observing that $z^n \in \mathcal{U}$ for all $n \geq 1$ we conclude that for all $n \geq 1$ the nonnonnegative random variable c^n belongs to the set $\mathcal{C}(z^n, B)$. Now, using the definition of c^n in conjunction with (30) we get that the inequality

$$E[U(c^n)] \geq \sum_{k=1}^{n} E[1_{A_k} U(x_0 + h^k)] \geq \sum_{k=1}^{n} E[1_{A_k} U(h^k)^+] \geq n\alpha$$

holds for all $n \geq 1$ and this implies

$$\limsup_{z\to\infty} \frac{v(z, B)}{z} \geq \limsup_{n\to\infty} \frac{E[U(c^n)]}{z^n} \geq \frac{\alpha}{z + \hat{u}(D)}.$$

Because the last term on the right hand side is strictly positive (recall that $z > \hat{u}(-B)$), the above expression contradicts the fact that $\ell = 0$ and therefore establishes the desired uniform integrability. The uniqueness of the optimal solution as well as the strict concavity of the value function now follows from the strict concavity of the agent's utility function.

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