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# CONVOLUTION OPERATORS AND HOMOMORPHISMS OF LOCALLY COMPACT GROUPS

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**Abstract**

Let $1<p<\infty$, let $G$ and $H$ be locally compact groups and let $\omega$ be a continuous homomorphism of $G$ into $H$. We prove, if $G$ is amenable, the existence of a linear contraction of the Banach algebra $CV_p(G)$ of the $p$-convolution operators on $G$ into $CV_p(H)$ which extends the usual definition of the image of a bounded measure by $\omega$. We also discuss the uniqueness of this linear contraction onto important subalgebras of $CV_p(G)$. Even if $G$ and $H$ are abelian, we obtain new results. Let $G_d$ denote the group $G$ provided with a discrete topology. As a corollary, we obtain, for every discrete measure, $|||\mu|||_{CV_p(G)} \le |||\mu|||_{CV_p(G_d)}$, for $G_d$ amenable. For arbitrary $G$, we also obtain $|||\mu|||_{CV_p(G_d)} \le |||\mu|||_{CV_p(G)}$. These inequalities were already known for $p=2$. The proofs presented in this paper avoid the use of the Hilbertian techniques which are not applicable to $p \neq 2$. Finally, for $G_d$ amenable, we construct a natural map of $CV_p(G)$ into $CV_p(G_d)$.

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## 1. Introduction

In 1965, de Leeuw [5] studied the transfer of $p$-multipliers from the circle $\mathbb{T}$ to $\mathbb{R}$ and from $\mathbb{R}_d$ to $\mathbb{R}$. These results were extended in part to locally compact abelian groups by Saeki [22], Lohoué [16–18] and Lust-Piquard [19]. The present paper investigates this problem for nonabelian locally compact groups.

Let $1 < p < \infty$, $\omega : G \to H$ be a continuous homomorphism of locally compact groups and $CV_p(G)$ be the set of all continuous operators on $L^p(G)$ commuting with left translation; they are called the $p$-convolution operators on $G$. Provided with the operator norm, denoted $|||\cdot|||_p$, $CV_p(G)$ is a Banach algebra. If $G$ is abelian, $CV_p(G)$ is isomorphic to the Banach algebra of all $p$-multipliers of $\widehat{G}$.

The first part of this paper is devoted to the transfer of convolution operators. We show in Theorem 3.1, if $G$ is amenable, that there is a linear contraction of $CV_p(G)$

into $CV_p(H)$ which generalizes the transfer of bounded measures. This map is unique for convolution operators with compact support (see Theorem 4.6). We give a global and a new point of view of the problem; our approach completely avoids the use of the structure theory of locally compact abelian groups and methods of Hilbert spaces. Moreover, we obtain new results even in the abelian case; we give a generalization of Reiter's theorem of relativization of the Beurling spectrum [21] in Scholium 5.5.

Theorem 3.1 gives us, if $G_d$ is amenable, a map of $CV_p(G_d)$ into $CV_p(G)$. In Theorem 6.6, in analogy to the Bohr compactification for $G$ abelian, we are able to construct a natural new map of $CV_p(G)$ into $CV_p(G_d)$, even if $G$ is nonabelian. On the way, we compare the operator norm of the discrete measures on $G$ and on $G_d$. Theorem 6.1 shows that $|||\mu|||_{CV_p(G_d)} \leq |||\mu|||_{CV_p(G)}$, for every discrete measure $\mu$ and that equality holds if $G_d$ is amenable. This result is already known for $p = 2$ (see [3, 4]), but the proof cannot be adapted to $p \neq 2$.

The ultraweak closure of the bounded measures in $CV_p(G)$ is denoted $PM_p(G)$ and called the Banach algebra of the $p$-pseudomeasures of $G$. If $p = 2$, $PM_2(G)$ is the von Neumann algebra $VN(G)$ of $G$. We recall that $PM_2(G) = VN(G) = CV_2(G)$. In this case, the study of the convolution operators is related to the theory of von Neumann algebras and Hilbert spaces. For example, the map $a \mapsto \lambda_G^2(\delta_a)$, where $\delta_a$ is the Dirac measure, is the left regular representation of $G$ on $L^2(G)$. These techniques are not applicable to $p \neq 2$.

## 2. Preliminaries

Let $1 < p < \infty$, $p' = p/(p-1)$, and let $G$, $H$ be two locally compact groups.

For any function $f$ on $G$, we define ${}_af(x) = f(ax)$, $f_a(x) = f(xa)$, $\check{f}(x) = f(x^{-1})$ and $\tilde{f}(x) = \overline{f(x^{-1})}$. For any measure $\mu$ on $G$, we define $\check{\mu}(f) = \mu(\check{f})$, $\bar{\mu}(f) = \overline{\mu(\bar{f})}$ and $\tilde{\mu}(f) = \overline{\mu(\tilde{f})}$. We define an isometric involution of $L^p(G)$ via $\tau_p\varphi = \Delta_G^{1/p'}\check{\varphi}$, where $\Delta_G$ denotes the modular function of $G$.

Let $M^1(G)$ denote the Banach algebra of the bounded measures of $G$. The map $\lambda_G^p$, defined via $\lambda_G^p(\mu)(\varphi) = \varphi \star \Delta_G^{1/p'}\check{\mu}$, where $\mu \in M^1(G)$ and $\varphi \in C_{oo}(G)$, is an injection of $M^1(G)$ into $CV_p(G)$.

We recall that $A_p(G)$ is the set of the bounded functions on $G$,

$$u = \sum_{n=1}^{\infty} \bar{f}_n \star \check{g}_n \quad \text{where } f_n \in L^p(G),\ g_n \in L^{p'}(G) \text{ and } \sum_{n=1}^{\infty} \|f\|_p \|g\|_{p'} < \infty,$$

$PM_p(G)$ is the dual of $A_p(G)$ and $CV_p(G) = PM_p(G)$, if $G$ is amenable or $p = 2$. Let $\langle \cdot, \cdot \rangle_{L^p(G),L^{p'}(G)}$ denote the duality of $L^p(G)$ and $L^{p'}(G)$. We recall that the duality of $A_p(G)$ and $PM_p(G)$ is given by

$$\langle u, T \rangle_{A_p(G),PM_p(G)} = \sum_{n=1}^{\infty} \overline{\langle T(\tau_p f_n), \tau_{p'} g_n \rangle}_{L^p(G),L^{p'}(G)},$$

where $u = \sum_{n=1}^{\infty} \bar{f}_n \star \check{g}_n$.

DEFINITION 2.1. For each $T \in PM_p(G)$, the support of $T$ is the set, denoted $\mathrm{supp}(T)$, of all $x \in G$ such that, for every neighborhood $V$ of $x$, there is $v \in A_p(G)$ such that $\mathrm{supp}(v) \subset V$ and $\langle v, T\rangle_{A_p(G),PM_p(G)} \neq 0$.

## 3. A transfer theorem for convolution operators

Our first main result is the following theorem.

THEOREM 3.1. *Let $1 < p < \infty$, let $G$, $H$ be two locally compact groups with $G$ amenable and let $\omega$ be a continuous homomorphism of $G$ into $H$. Then there is a linear contraction*

$$\omega : CV_p(G) \to CV_p(H)$$

*which satisfies*

$$\omega(\lambda_G^p(\mu)) = \lambda_H^p(\omega(\mu)) \quad \textit{for each bounded measure } \mu \textit{ of } G.$$

To prove this theorem, we need the following preliminaries.

Let $\omega : G \to H$ be a continuous homomorphism. For each $T \in CV_p(G)$, $f$, $g \in C_{oo}(G)$, $\varphi \in L^p(H)$ and $\psi \in L^{p'}(H)$, we consider the function of $H$

$$h \mapsto \langle T(\tau_p(f((\tau_p\varphi)_h \circ \omega))), \tau_{p'}(g((\tau_{p'}\psi)_h \circ \omega))\rangle_{L^p(G),L^{p'}(G)}.$$

This function is integrable and continuous on $H$ with its $L^1$-norm bounded by $|||T|||_p\|f\|_p\|g\|_{p'}\|\varphi\|_p\|\psi\|_{p'}$. Then for each $T \in CV_p(G)$ and $f$, $g \in C_{oo}(G)$, there is a unique $p$-convolution operator on $H$, denoted $\omega_{f,g}(T)$, such that, for all $(\varphi, \psi) \in L^p(H) \times L^{p'}(H)$,

$$\begin{aligned}\langle \omega_{f,g}(T)\varphi, \psi\rangle_{L^p(H),L^{p'}(H)} \\ = \int_H \langle T(\tau_p(f((\tau_p\varphi)_h \circ \omega))), \tau_{p'}(g((\tau_{p'}\psi)_h \circ \omega))\rangle_{L^p(G),L^{p'}(G)}\, dh.\end{aligned}$$

PROPOSITION 3.2. *Let $G$ and $H$ be two locally compact groups (not necessary amenable) and $\omega : G \to H$ be a continuous homomorphism. Let $f$, $g \in C_{oo}(G)$. Then $\omega_{f,g}$ is a linear map of $CV_p(G)$ into $CV_p(H)$ and $\|\omega_{f,g}\| \le \|f\|_p\ \|g\|_{p'}$. Moreover, for each $\mu \in M^1(G)$ and $f$, $g \in C_{oo}(G)$,*

$$\langle \omega_{f,g}(\lambda_G^p(\mu))\varphi, \psi\rangle_{L^p(H),L^{p'}(H)} = \overline{\tilde{\mu}(\bar{f} \star \check{g}(\overline{\tau_p\varphi} \star (\tau_{p'}\psi)\check{}) \circ \omega)}.$$

We can immediately compare this result with

$$\langle \lambda_H^p(\omega(\mu))\varphi, \psi\rangle_{L^p(H),L^{p'}(H)} = \overline{\tilde{\mu}((\overline{\tau_p\varphi} \star (\tau_{p'}\psi)\check{}) \circ \omega)},$$

and see that, if $\bar{f} \star \check{g}$ is close to 1, $\lambda_H^p(\omega(\mu))$ is close to $\omega_{f,g}(\lambda_G^p(\mu))$.

REMARK 3.3. The special cases where $\omega$ is the inclusion of a closed subgroup or the projection on a quotient were already treated in [1, 2, 6–9]. Combining these two cases, it is possible to treat open continuous homomorphisms. The study of a general continuous homomorphism requires new ideas.

PROOF OF THEOREM 3.1.
Let $f, g \in C_{oo}(G)$. For all $T \in CV_p(G)$ and $(\varphi, \psi) \in L^p(H) \times L^{p'}(H)$, we define

$$\Omega_{f,g}(T, \varphi, \psi) = \langle \omega_{f,g}(T)\varphi, \psi \rangle_{L^p(H), L^{p'}(H)}.$$

In fact, $\Omega_{f,g}$ is a continuous form on $CV_p(G) \times L^p(H) \times L^{p'}(H)$ which is bilinear on the first two factors and conjugate linear on the last. Let $\mathcal{B}$ denote the set of these forms provided with the weak topology of duality with $CV_p(G) \times L^p(H) \times L^{p'}(H)$.

For every compact $K \subset G$ and $\varepsilon > 0$, we define

$$\mathcal{U}_{K,\varepsilon} = \{U \subset G : U \text{ compact}, m(U) > 0, m(xU \triangle U) < \varepsilon m(U)\ \forall x \in K\}$$
$$\mathcal{A}_{K,\varepsilon} = \{\Omega_{f,g} : f = m(U)^{-1/p} 1_U,\ g = m(U)^{-1/p'} 1_U,\ U \in \mathcal{U}_{K,\varepsilon}\}.$$

By the Banach–Alaoglu theorem,

$$\mathcal{S} = \{F \in \mathcal{B} : |F(T, \varphi, \psi)| \le |||T|||_p \|\varphi\|_p \|\psi\|_{p'},$$
$$\text{for all } (T, \varphi, \psi) \in CV_p(G) \times L^p(H) \times L^{p'}(H)\}$$

is a compact subset of $\mathcal{B}$. Since $G$ is amenable, it satisfies the property $\underline{(F)}$ of Følner [20, Theorem 7.3], so the $\mathcal{U}_{K,\varepsilon}$ are all nonempty. Then, the family $\{\overline{\mathcal{A}_{K,\varepsilon}}\}$ (where $\overline{\mathcal{A}_{K,\varepsilon}}$ denotes the weak closure of $\mathcal{A}_{K,\varepsilon}$) have the property of finite intersection. However, each $\mathcal{A}_{K,\varepsilon} \subset \mathcal{S}$ and $\mathcal{S}$ is a compact set, so

$$\bigcap_{\substack{K \subset G \text{ compact}, \\ \varepsilon > 0}} \overline{\mathcal{A}_{K,\varepsilon}} \ne \emptyset.$$

Let

$$\Omega \in \bigcap_{\substack{K \subset G \text{ compact}, \\ \varepsilon > 0}} \overline{\mathcal{A}_{K,\varepsilon}}.$$

There is a unique continuous linear operator $\omega(T)$ on $L^p(G)$ such that, for all $(\varphi, \psi) \in L^p(H) \times L^{p'}(H)$,

$$\langle \omega(T)\varphi, \psi \rangle_{L^p(H), L^{p'}(H)} = \Omega(T, \varphi, \psi).$$

By construction, $\omega(T) \in CV_p(H)$ and $\omega$ is a contraction.

Let $\mu \in M^1(G)$. We prove that

$$\omega(\lambda_G^p(\mu)) = \lambda_H^p(\omega(\mu)).$$

Let $(\varphi, \psi) \in L^p(H) \times L^{p'}(H)$ and $\varepsilon > 0$. We consider

$$0 < \delta < \varepsilon[1 + \omega(|\mu|)(\Delta_H^{1/p}|\varphi \star \tilde{\psi}|) + 2\|(\Delta_H^{1/p}\varphi \star \tilde{\psi}) \circ \omega\|_\infty]^{-1}.$$

There is a compact subset $K_\delta \subset G$ such that $|\mu|(G \setminus K_\delta) < \delta$. By definition of $\Omega$, there is a compact subset $U \in \mathcal{U}_{K_\delta,\delta}$ such that

$$|\Omega_{f,g}(\lambda_G^p(\mu), \varphi, \psi) - \Omega(\lambda_G^p(\mu), \varphi, \psi)| < \frac{\varepsilon}{2},$$

where $f = m(U)^{-1/p}1_U$ and $g = m(U)^{-1/p'}1_U$. In fact, for all $x \in K_\delta^{-1}$,

$$0 \le 1 - \frac{m(x^{-1}U \cap U)}{m(U)} < \frac{\delta}{2}.$$

Let $(\varphi, \psi) \in L^p(H) \times L^{p'}(H)$. On the one hand,

$$\begin{aligned}\bigg|\int_{K_\delta}\bigg(1 - \frac{m(x^{-1}U \cap U)}{m(U)}\bigg)\Delta_H^{1/p}(\omega(x))\varphi \star \tilde{\psi}(\omega(x))\,d\mu(x)\bigg| \\ \le \frac{\delta}{2}\,\omega(|\mu|)(\Delta_H^{1/p}|\varphi \star \tilde{\psi}|).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\bigg|\int_{G\setminus K_\delta}\bigg(1 - \frac{m(x^{-1}U \cap U)}{m(U)}\bigg)\Delta_H^{1/p}(\omega(x))\varphi \star \tilde{\psi}(\omega(x))\,d\mu(x)\bigg| \\ &\quad\le \|\Delta_H^{1/p} \circ \omega\,\varphi \star \tilde{\psi} \circ \omega\|_\infty\,|\mu|(G \setminus K_\delta) \le \|(\Delta_H^{1/p}\varphi \star \tilde{\psi}) \circ \omega\|_\infty\delta.\end{aligned}$$

Finally,

$$\begin{aligned}&|\langle\omega(\lambda_G^p(\mu))\varphi, \psi\rangle_{L^p(H),L^{p'}(H)} - \langle\lambda_H^p(\omega(\mu))\varphi, \psi\rangle_{L^p(H),L^{p'}(H)}| \\ &\quad\le |\langle\omega(\lambda_G^p(\mu))\varphi, \psi\rangle_{L^p(H),L^{p'}(H)} - \langle\omega_{f,g}(\lambda_G^p(\mu))\varphi, \psi\rangle_{L^p(H),L^{p'}(H)}| \\ &\qquad + |\langle\omega_{f,g}(\lambda_G^p(\mu))\varphi, \psi\rangle_{L^p(H),L^{p'}(H)} - \langle\lambda_H^p(\omega(\mu))\varphi, \psi\rangle_{L^p(H),L^{p'}(H)}| \\ &\quad\le \frac{\varepsilon}{2} + \bigg|\int_G\bigg(1 - \frac{m(x^{-1}U \cap U)}{m(U)}\bigg)\Delta_H^{1/p}(\omega(x))\varphi \star \tilde{\psi}(\omega(x))\,d\mu(x)\bigg| \\ &\quad< \frac{\varepsilon}{2} + \frac{\delta}{2}\,\omega(|\mu|)(\Delta_H^{1/p}|\varphi \star \tilde{\psi}|) + \|(\Delta_H^{1/p}\varphi \star \tilde{\psi}) \circ \omega\|_\infty\,\delta < \varepsilon.\end{aligned}$$

□

REMARK 3.4. Instead of Følner's property, we could use the property $(P_p)$ of Reiter [20, Proposition 6.12]. It is sufficient to consider the set

$$\begin{aligned}\mathcal{R}_{K,\varepsilon} = \{\Omega_{f,g} : f, g > 0, \|f\|_p = \|g\|_{p'} = 1, \\ \|_a f - f\|_p < \varepsilon \text{ and } \|_a g - g\|_{p'} < \varepsilon \text{ for all } a \in K\}.\end{aligned}$$

With the same arguments, we obtain that

$$\bigcap_{\substack{K\subset G \text{ compact},\\ \varepsilon>0}} \overline{\mathcal{R}_{K,\varepsilon}} \neq \emptyset.$$

REMARK 3.5.

(1) The definition of the convolution operator $\omega_{f,g}(T)$ does not require the amenability of $G$.
(2) Using duality techniques of Herz [11, 12], one can give a shorter proof of Theorem 3.1. We have presented the above proof as it uses more basic ideas. We use duality arguments in the next section.

## 4. Image of a pseudomeasure and the $A_p$ algebras

We show now that $\omega(T)$ is uniquely defined for $T$ in the norm closure of the set of all compactly supported convolution operators. This Banach algebra is denoted $cv_p(G)$. We recall that $cv_2(G) = C_u^b(\widehat{G})$, since $G$ is abelian.

For $u \in A_p(G)$ and $T \in PM_p(G)$, it is useful to define $\omega_u(T)$ by

$$\omega_u(T) = \sum_{n=1}^{\infty} \omega_{f_n,g_n}(T) \quad \text{where } u = \sum_{n=1}^{\infty} \bar{f}_n \star \check{g}_n.$$

The map $\omega_u$ is well defined because

$$\sum_{n=1}^{\infty} \overline{\langle \omega_{f_n,g_n}(T)(\tau_p\varphi), \tau_{p'}\psi\rangle}_{L^p(H),L^{p'}(H)} = \left\langle (\bar{\varphi} \star \check{\psi}) \circ \omega \sum_{n=1}^{\infty} \bar{f}_n \star \check{g}_n, T \right\rangle_{A_p(G),PM_p(G)}.$$

The following proposition is similar to Proposition 3.2.

PROPOSITION 4.1. *Let $1 < p < \infty$ and $u \in A_p(G)$. Then, $\omega_u$ is a linear map of $PM_p(G)$ into $PM_p(H)$ such that $|||\omega_u(T)|||_p \le |||T|||_p \, \|u\|_{A_p}$.*

REMARK 4.2. Let us assume that $G$ is amenable. Theorem 3.1 implies that, for every $T \in PM_p(G)$, $\varepsilon > 0$, $v \in A_p(H)$, there is $u \in A_p(G)$ such that

$$|\langle v, \omega(T)\rangle_{A_p(G),PM_p(G)} - \langle v, \omega_u(T)\rangle_{A_p(G),PM_p(G)}| < \varepsilon.$$

Let $MA_p$ denote the set of the multipliers of $A_p$ (that is, $v \in MA_p$, if $vu \in A_p$, for all $u \in A_p$). It is well known that $MA_p(G)$ multiplies $PM_p(G)$ in the sense of

$$\langle v, uT\rangle_{A_p(G),PM_p(G)} = \langle uv, T\rangle_{A_p(G),PM_p(G)},$$

for all $u \in MA_p(G)$, $v \in A_p(G)$ and $T \in PM_p(G)$. We recall that $\omega(u) \in MA_p(H)$, for all $u \in MA_p(G)$, see [13].

PROPOSITION 4.3. *Let $T \in PM_p(G)$ and $u \in MA_p(H)$. If $G$ is amenable, then*

$$\omega((u \circ \omega)T) = u\,\omega(T).$$

PROOF. Let $\varepsilon > 0$ and $w \in A_p(H)$.

There is $v \in A_p(G)$ such that

$$|\langle w, \omega((u \circ \omega)T)\rangle_{A_p(H),PM_p(H)} - \langle w, \omega_v((u \circ \omega)T)\rangle_{A_p(H),PM_p(H)}| < \frac{\varepsilon}{2}$$

and

$$|\langle uw, \omega(T)\rangle_{A_p(H),PM_p(H)} - \langle uw, \omega_v(T)\rangle_{A_p(H),PM_p(H)}| < \frac{\varepsilon}{2}.$$

However, $\langle w, \omega_v((u \circ \omega)T)\rangle_{A_p(H),PM_p(H)} = \langle uw, \omega_v(T)\rangle_{A_p(H),PM_p(H)}$. □

LEMMA 4.4. *Let $T \in PM_p(G)$ and $u \in A_p(G)$. If $h \in \operatorname{supp}(\omega_u(T))$, then for every neighborhood $V$ of $h$, there is $v \in A_p(G)$ with $\operatorname{supp}(v) \subset \omega^{-1}(V)$ such that*

$$\langle v, T\rangle_{A_p(G),PM_p(G)} \neq 0.$$

THEOREM 4.5. *Let $T \in PM_p(G)$. If $G$ is amenable, then*

$$\operatorname{supp}(\omega(T)) \subset \overline{\omega(\operatorname{supp}(T))}.$$

PROOF. Let $u \in A_p(G)$. First, we prove that $\operatorname{supp}(\omega_u(T)) \subset \overline{\omega(\operatorname{supp}(T))}$.

Let $h \in \operatorname{supp}(\omega_u(T))$ and suppose $h \notin \overline{\omega(\operatorname{supp}(T))}$. Then there exists a closed neighborhood $V$ of $h$ in $H$ such that

$$V \cap \omega(\operatorname{supp}(T)) = \emptyset.$$

Let $v \in A_p(H)$ with $\operatorname{supp}(v) \subset V$. For each $x \in G$ with $((v \circ \omega)u)(x) \neq 0$, we have $x \in \omega^{-1}(V)$, so $\operatorname{supp}((v \circ \omega)u) \subset \omega^{-1}(V)$. However, $\omega^{-1}(V) \cap \operatorname{supp}(T) = \emptyset$. Then, $((v \circ \omega)u)T = 0$, and by the amenability of $G$,

$$\langle (v \circ \omega)u, T\rangle_{A_p(G),PM_p(G)} = 0,$$

which contradicts Lemma 4.4.

Finally, we prove that

$$\operatorname{supp}(\omega(T)) \subset \overline{\bigcap_{u \in A_p(G)} \operatorname{supp}(\omega_u(T))}.$$

Let $h_0 \in \operatorname{supp}(\omega(T))$. Suppose

$$h_0 \notin \overline{\bigcap_{u \in A_p(G)} \operatorname{supp}(\omega_u(T))}.$$

Then there exists a closed neighborhood $V_0$ of $h_0$ in $H$ such that, for all $u \in A_p(G)$, $V_0 \cap \operatorname{supp}(\omega_u(T)) = \emptyset$. Let $v \in A_p(H)$ with $\operatorname{supp}(v) \subset V_0$. For each $u \in A_p(G)$, $\operatorname{supp}(v) \cap \operatorname{supp}(\omega_u(T)) = \emptyset$, then $v\,\omega_u(T) = 0$ and by the amenability of $G$,

$$\langle v, \omega_u(T)\rangle_{A_p(H),PM_p(H)} = 0.$$

It follows that

$$\langle v, \omega(T)\rangle_{A_p(H),PM_p(H)} = 0,$$

which contradicts Lemma 4.4. □

We now want to prove that the transfer mapping is uniquely defined on a larger class of convolution operators, notably on $cv_p(G)$. We recall that, if $G$ is amenable, then $cv_p(G) = A_p(G)PM_p(G)$, as a direct consequence of the Cohen–Hewitt theorem [14, Ch. VIII, Paragraph 32].

THEOREM 4.6. *Let $T \in PM_p(G)$ and $u \in A_p(G)$. If $G$ is amenable, then*

$$\omega(uT) = \omega_u(T).$$

*In fact, there is a* unique *linear contraction $\omega : cv_p(G) \to cv_p(H)$ which generalizes the transfer of bounded measures.*

PROOF. Let $T \in PM_p(G)$, $u \in A_p(G)$, $v \in A_p(H)$ and $\varepsilon > 0$. There is $w \in A_p(G)$ such that

$$|\langle v, \omega(uT)\rangle_{A_p(H),PM_p(H)} - \langle v, \omega_w(uT)\rangle_{A_p(H),PM_p(H)}| < \frac{\varepsilon}{2}$$

and

$$|\langle v \circ \omega\, u, T\rangle_{A_p(G),PM_p(G)} - \langle v \circ \omega\, u\, w, T\rangle_{A_p(G),PM_p(G)}| < \frac{\varepsilon}{2}.$$

However,

$$\begin{aligned}\langle v, \omega_w(uT)\rangle_{A_p(H),PM_p(H)} &= \langle v \circ \omega\, w, uT\rangle_{A_p(G),PM_p(G)}\\ &= \langle v \circ \omega\, u\, w, T\rangle_{A_p(G),PM_p(G)}.\end{aligned}$$

Then,

$$\begin{aligned}\langle v, \omega(uT)\rangle_{A_p(H),PM_p(H)} &= \langle v \circ \omega\, u, T\rangle_{A_p(G),PM_p(G)}\\ &= \langle v, \omega_u(T)\rangle_{A_p(H),PM_p(H)}.\end{aligned}$$

Finally, we prove that $\omega(T) \in cv_p(G)$. There is a sequence $(T_n)_{n=1}^{\infty}$ of convolution operators with compact support such that $|||T_n - T|||_p \to 0$, and $(K_n)_{n=1}^{\infty}$ is a sequence of compact subsets of $G$ with $\operatorname{supp}(T_n) \subset K_n$. For each $n \in \mathbb{N}$, $\operatorname{supp}(\omega(T_n)) \subset \omega(K_n)$. However, $\omega : CV_p(G) \to CV_p(H)$ is ultraweak continuous, then $|||\cdot|||_p$-continuous. So

$$\lim_{n\to\infty} \omega(T_n) = \omega\Big(\lim_{n\to\infty} T_n\Big) = \omega(T).$$ □

EXAMPLE 4.7. Let $H$ be a closed amenable subgroup of $G$ and let $\omega = i : H \to G$ be the canonical inclusion. For all $T \in PM_p(H)$ and $v \in A_p(G)$,

$$\langle v, i(T)\rangle_{A_p(G),PM_p(G)} = \langle \mathrm{Res}_H v, T\rangle_{A_p(H),PM_p(H)}.$$

Derighetti obtained this result without supposing the amenability of the subgroup $H$ (see [8, Theorem 2, p. 76]). However, he used techniques which cannot be applied to arbitrary continuous homomorphisms.

EXAMPLE 4.8. Let $G$ be an amenable locally compact group and $\omega : G \to \{e\}$ be the trivial homomorphism. Then there is a linear contraction

$$\omega : CV_p(G) \to \mathbb{C}$$

with the following properties:

(1) $\omega(\lambda_G^p(\mu)) = \mu(G)$ for each bounded measure $\mu$ of $G$;
(2) $\omega(uT) = \langle u, T\rangle_{A_p(G),PM_p(G)}$ for each $u \in A_p(G)$.

In fact, this defines a kind of integral on $CV_p(G)$!

EXAMPLE 4.9. Let $G$ be an arbitrary Lie group. Then, for each $X$ in its Lie algebra, there is a continuous homomorphism of $\mathbb{R}$ into $G$ defined by $t \mapsto \exp(tX)$. Hence, we are able to transfer every $T \in CV_p(\mathbb{R})$ into $CV_p(G)$.

## 5. The abelian case

The aim of this section is to compute the Fourier transform of $\omega(T)$.

Let $G$ and $H$ be two locally compact *abelian* groups and $\omega : G \to H$ be a continuous homomorphism. Here $A(G)$ denotes the Fourier algebra of $G$ (we recall that $A(G) = A_2(G)$) and $\widehat{G}$ be the character group of $G$. We denote by $\varepsilon_G : G \to \widehat{\widehat{G}}$ the canonical isomorphism defined by $\varepsilon_G(x)(\chi) = \chi(x)$, for all $x \in G$ and $\chi \in \widehat{G}$. We define an isometric isomorphism $\Phi_{\widehat{G}} : L^1(\widehat{G}) \to A_2(G)$ by

$$\Phi_{\widehat{G}}(f)(x) = \int_{\widehat{G}} f(\chi)\varepsilon_G(x)(\chi)\, d\chi,$$

for all $x \in G$ and the Fourier transform $\hat{}: L^1(\widehat{G}) \to A_2(\widehat{\widehat{G}})$ by

$$\hat{f}(\xi) = \int_{\widehat{G}} f(\chi)\overline{\xi(\chi)}\, d\chi,$$

for all $\xi \in \widehat{\widehat{G}}$. Let $\mathcal{F} : L^2(\widehat{G}) \to L^2(\widehat{\widehat{G}})$ denote the extension of $\hat{}$ on $L^2(\widehat{G})$.

Let $\hat{\omega} : \widehat{H} \to \widehat{G}$ denote the dual homomorphism defined by $\hat{\omega}(\chi') = \chi' \circ \omega$, for all $\chi' \in \widehat{H}$. For each $T \in CV_2(G)$, $\widehat{T}$ denotes the Fourier transform of $T$, that is the unique function of $L^\infty(\widehat{G})$ such that, for all $\varphi, \psi \in L^2(\widehat{G})$,

$$\langle T\varphi, \psi\rangle_{L^2(G),L^2(G)} = \langle \widehat{T}\, \mathcal{F}(\varphi), \mathcal{F}(\psi)\rangle_{L^2(\widehat{G}),L^2(\widehat{G})}.$$

Let $1 < p < \infty$. We define a contractive monomorphism $\alpha_p : CV_p(G) \to CV_2(G)$ such that, for all $\varphi \in L^2(G) \cap L^p(G)$, $\alpha_p(T)(\varphi) = T(\varphi)$. For $T \in CV_p(G)$, the Fourier transform of $T$ is defined by

$$\widehat{T} = \widehat{\alpha_p(T)}. \tag{5.1}$$

From these definitions we have the following lemma.

LEMMA 5.1. *Let* $1 < p < \infty$, $u \in A(G)$ *and* $T \in CV_p(G)$. *Then*

$$\omega_u(\alpha_p(T)) = \omega_u(T)$$

*and*

$$\widehat{\omega_u(T)} = (\widehat{T} \star \widetilde{\Phi^{-1}(u)}) \circ \hat{\omega}.$$

THEOREM 5.2. *Let* $T \in PM_p(G)$ *with* $\widehat{T}$ *continuous on* $\widehat{G}$. *Then,*

$$\widehat{\omega(T)} = \widehat{T} \circ \hat{\omega}.$$

PROOF. First, we consider $S = \alpha_p(T) \in CV_2(G)$. Let $\varepsilon > 0$ and $f \in L^1(\widehat{H})$. By hypothesis, $\widehat{S}$ is a continuous function on $\widehat{G}$. So, for all $\chi \in \widehat{G}$, there is a compact neighborhood $C$ of $e \in \widehat{G}$ such that

$$|\widehat{S}(\chi\chi') - \widehat{S}(\chi)| < \frac{\varepsilon}{4(1 + \|f\|_1)},$$

for all $\chi' \in C$.

There is $\delta > 0$ and a compact $K \subset G$ such that, for all $U \in \mathcal{U}_{K^{-1},\delta}$,

$$\int_{\widehat{G}\setminus C} \Phi^{-1}(v)(\chi)\, d\chi < \frac{\varepsilon}{8(1 + \|\widehat{S}\|_\infty)(1 + \|f\|_1)}$$

and

$$|\langle u, \omega(S)\rangle_{A_p(H),PM_p(H)} - \langle u, \omega_v(S)\rangle_{A_p(H),PM_p(H)}| < \delta,$$

where $v = m(U)^{-1}\, 1_U \star \check{1}_U \in A(G)$. Then,

$$\|\widehat{S} \star \widetilde{\Phi^{-1}(v)} - \widehat{S}\|_\infty < \frac{\varepsilon}{2(1 + \|f\|_1)}.$$

On the one hand,

$$\begin{aligned} &|\langle f, \widehat{\omega(S)}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})} - \langle f, \widehat{\omega_v(S)}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})}| \\ &\quad = |\langle \Phi(f), \omega(S)\rangle_{A(H),PM_2(G)} - \langle \Phi(f), \omega_v(S)\rangle_{A(H),PM_2(G)}| < \frac{\varepsilon}{2}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
&|\langle f, \widehat{\omega_v(S)}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})} - \langle f, \widehat{S}\circ\hat{\omega}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})}| \\
&\quad = |\langle f, (\widehat{S}\star\widetilde{\Phi^{-1}(v)})\circ\hat{\omega}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})} - \langle f, \widehat{S}\circ\hat{\omega}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})}| \\
&\quad = |\langle f, (\widehat{S}\star\widetilde{\Phi^{-1}(v)} - \widehat{S})\circ\hat{\omega}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})}| < \|f\|_1 \frac{\varepsilon}{2(1+\|f\|_1)} < \frac{\varepsilon}{2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
&|\langle f, \widehat{\omega(S)}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})} - \langle f, \widehat{S}\circ\hat{\omega}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})}| \\
&\quad \le |\langle f, \widehat{\omega(S)}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})} - \langle f, \widehat{\omega_v(S)}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})}| \\
&\qquad + |\langle f, \widehat{\omega_v(S)}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})} - \langle f, \widehat{S}\circ\hat{\omega}\rangle_{L^1(\widehat{H}),L^\infty(\widehat{H})}| < \varepsilon
\end{aligned}$$

and

$$\widehat{\omega(S)} = \widehat{S}\circ\hat{\omega}.$$

We conclude by applying (5.1). □

REMARK 5.3. Theorem 5.2 was previously proved by Lohoué [17, Theorem I.1] and [15]. Nonabelian methods allow us to give a new proof.

REMARK 5.4. Let $G$ be a locally compact abelian group and consider the homomorphism of Example 4.8,

$$\omega : G \to \{e\}.$$

Then, for each $T \in CV_p(G)$ with $\widehat{T}$ continuous,

$$\omega(\widehat{T}) = \widehat{T}(1).$$

Let $\varphi \in L^\infty(G)$. We recall that the spectrum of $\varphi$ is the set

$$\mathrm{sp}(\varphi) = \{\chi \in \widehat{G} : \widehat{f}(\chi) = 0 \text{ for all } f \in L^1(G) \text{ with } f\star\varphi = 0\},$$

and that

$$\varepsilon_G(\mathrm{supp}(T)) = (\mathrm{sp}(\widehat{T}))^{-1}.$$

SCHOLIUM 5.5. *Let $T \in CV_p(G)$ with $\widehat{T}$ continuous. Then,*

$$\mathrm{sp}(\widehat{T}\circ\widehat{\omega}) \subset \overline{\widehat{\widehat{\omega}}(\mathrm{sp}(\widehat{T}))}.$$

PROOF. By Theorem 4.5, we have $\mathrm{supp}(\omega(T)) \subset \overline{\omega(\mathrm{supp}(T))}$ and then

$$\varepsilon_G(\mathrm{supp}(\omega(T))) \subset \overline{\widehat{\widehat{\omega}}(\varepsilon_G(\mathrm{supp}(T)))}.$$

By Theorem 5.2,

$$\mathrm{sp}(\widehat{T}\circ\widehat{\omega}) \subset (\overline{\widehat{\widehat{\omega}}((\mathrm{sp}(\widehat{T}))^{-1})})^{-1} \subset \overline{\widehat{\widehat{\omega}}((\mathrm{sp}(\widehat{T})))}.$$ □

REMARK 5.6. In [21, Theorem 7.2.2, p. 200], Reiter proves a result called 'relativisation of the spectrum'. It is, in fact, a particular case of the Scholium 5.5 where $\widehat{H}$ is a closed subgroup of $\widehat{G}$ and $\hat{\omega}$ is the inclusion.

## 6. Relations between $CV_p(G)$ and $CV_p(G_d)$

We know that deep relations exist between the harmonic analysis of $G$ and $G_d$. In [10, 12], Eymard and Herz investigated the relationship between $B(G)$ and $B(G_d) \cap C(G)$. In this section, we study the relationship between $CV_p(G)$ and $CV_p(G_d)$. More precisely, we construct a new map of $CV_p(G)$ into $CV_p(G_d)$, for $G_d$ amenable.

First, we give results about the operator norm of discrete measures. For each sequence $(c_n)_{n=1}^{\infty} \in \ell^1$ and $(a_n)_{n=1}^{\infty}$ on $G$, we consider the measure $\mu = \sum_{n=1}^{\infty} c_n \delta_{a_n}$, where $\delta_a$ is the Dirac measure on $a$. Here $\mu$ is a bounded measure on both $G$ and on $G_d$ with $\omega(\mu) = \mu$. All of these measures are called discrete measures of $G$.

THEOREM 6.1. *Let $1 < p < \infty$, $G$ be a locally compact group and let $\mu$ be a discrete measure of $G$. Then,*

$$|||\lambda_{G_d}^p(\mu)|||_p \leq |||\lambda_G^p(\mu)|||_p.$$

*Moreover, if $G_d$ is amenable,*

$$|||\lambda_{G_d}^p(\mu)|||_p = |||\lambda_G^p(\mu)|||_p.$$

The proof of the first inequality is based on the following construction and the second is a corollary of Theorem 3.1.

DEFINITION 6.2. Let $W$ be a relatively compact neighborhood of $e$ in $G$. For each $k \in C_{oo}(G_d)$, we define

$$T_W^p(k) = m(W)^{-1/p} \sum_{x \in G} k(x)_{x^{-1}}(1_W).$$

It is straightforward to prove the following properties:

(1) $T_W^p : C_{oo}(G_d) \to L^p(G)$;
(2) $\|T_W^p(k)\|_p \leq \|k\|_1$;
(3) $T_W^p({}_a k) = {}_a(T_W^p(k))$ for all $a \in G$.

The second property can be improved on, as follows.

LEMMA 6.3. *Let $k \in C_{oo}(G_d)$ with $\mathrm{supp}(k) = \{x_1, \ldots, x_n\}$ and let $W$ be a relatively compact neighborhood of $e$ in $G$ such that $x_i W \cap x_j W = \emptyset$, for each $1 \leq i, j \leq n$ with $i \neq j$. Then*

$$\|T_W^p(k)\|_p = \|k\|_p.$$

PROOF. For each $y \in G$, there is $j_y \in \{1, \dots, n\}$ such that $x_{j_y}^{-1} y \in W$. Then

$$\left|\sum_{j=1}^{n} k(x_j) 1_W(x_j^{-1} y)\right|^p = \sum_{j=1}^{n} |k(x_j)|^p \, 1_W(x_j^{-1} y) \quad \text{and}$$

$$\|T_W^p(k)\|_p^p = m(W)^{-1} \sum_{j=1}^{n} |k(x_j)|^p \int_G 1_W(y)\, dy = \|k\|_p. \qquad \square$$

LEMMA 6.4. *Let $k, l \in C_{oo}(G_d)$ and let $\mu$ be a bounded measure on $G$ with finite support (that is, $\mu = \sum_{i=1}^{n} c_i \delta_{a_i}$, where $c_i \in \mathbb{C}$ and $a_i \in G$). Then there exists a neighborhood $W$ of $e$ in $G$ such that*

$$\langle \overline{k} \star \check{l}, \lambda_{G_d}^p(\mu)\rangle_{A_p(G_d), PM_p(G_d)} = \langle (T_W^p(\overline{k})) \star (T_W^{p'}(l))\check{}, \lambda_G^p(\mu)\rangle_{A_p(G), PM_p(G)}.$$

PROOF. Suppose that $\mu = \delta_a$, for any $a \in G$. Let $\operatorname{supp}(k) = \{x_1, \dots, x_n\}$ and $\operatorname{supp}(l) = \{y_1, \dots, y_m\}$.

We define $E = \{(i, j) \in \mathbb{N}_n \times \mathbb{N}_m : a x_i = y_j\}$ and consider $W$ a neighborhood of $e$ such that $(x_i^{-1} a^{-1} y_j) W \cap W = \emptyset$. Then,

$$\begin{aligned}
&\langle (T_W^p(\overline{k})) \star (T_W^{p'}(l))\check{}, \lambda_G^p(\mu)\rangle_{A_p(G), PM_p(G)} \\
&\quad = m(W)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{k}(x_i) l(y_j) \int_G 1_W(x_i^{-1} a^{-1} y_j y) 1_W(y)\, dy \\
&\quad = \sum_{(i,j)\in E} \overline{k}(a^{-1} y_j) l(y_j) = \sum_{j=1}^{m} \overline{k}(a^{-1} y_j) l(y_j) \\
&\quad = \langle \overline{k} \star \check{l}, \lambda_{G_d}^p(\mu)\rangle_{A_p(G_d), PM_p(G_d)}.
\end{aligned}$$

The result now follows by linearity. $\square$

PROOF OF THEOREM 6.1. We prove that $|||\lambda_{G_d}^p(\mu)|||_p \le |||\lambda_G^p(\mu)|||_p$.

Let $r \in C_{oo}(G_d)$ with $\|r\|_1 \le 1$. We define $f = r^{1/p}$ and $g = r^{1/p'}$. Let $\nu$ be a bounded measure with finite support. There is a neighborhood $W$ of $e$ in $G$ such that

$$\begin{aligned}
&\langle \overline{\tau_p f} \star (\tau_p g)\check{}, \lambda_{G_d}^p(\nu)\rangle_{A_p(G_d), PM_p(G_d)} \\
&\quad = \langle T_W^p(\overline{\tau_p f}) \star T_W^{p'}((\tau_p g)\check{}), \lambda_G^p(\nu)\rangle_{A_p(G), PM_p(G)}.
\end{aligned}$$

Then

$$\begin{aligned}
&|\langle \lambda_{G_d}^p(\nu) f, g\rangle_{\ell^p(G), \ell^{p'}(G)}| \\
&\quad = |\langle \lambda_G^p(\nu)(\tau_p(T_W^p(\tau_p f))), \tau_{p'}(T_W^{p'}(\tau_{p'} g))\rangle_{L^p(G), L^{p'}(G)}| \\
&\quad \le |||\lambda_G^p(\nu)|||_p \, \|T_W^p(\tau_p f)\|_p \, \|T_W^{p'}(\tau_{p'} g)\|_{p'} \le |||\lambda_G^p(\nu)|||_p \, \|f\|_p \, \|g\|_{p'}.
\end{aligned}$$

Finally, from $\|f\|_p \leq 1$ and $\|g\|_{p'} \leq 1$ , we obtain $|||\lambda^p_{G_d}(\nu)|||_p \leq |||\lambda^p_G(\nu)|||_p$.

There is a $(\nu_n)_{n=1}^{\infty}$ sequence of bounded measures of $G$ with finite support such that $\lim \|\nu_n - \mu\| = 0$.

$$\begin{aligned}
|||\lambda^p_{G_d}(\mu)|||_p &\leq |||\lambda^p_{G_d}(\mu) - \lambda^p_{G_d}(\nu_n)|||_p + |||\lambda^p_{G_d}(\nu_n)|||_p \\
&\leq |||\lambda^p_{G_d}(\mu - \nu_n)|||_p + |||\lambda^p_G(\nu_n)|||_p \\
&\leq \|\mu - \nu_n\| + |||\lambda^p_G(\nu_n) - \lambda^p_G(\mu)|||_p + |||\lambda^p_G(\mu)|||_p \\
&\leq 2\|\mu - \nu_n\| + |||\lambda^p_G(\mu)|||_p.
\end{aligned}$$

Assume that $G_d$ is amenable. The inequality $|||\lambda^p_{G_d}(\mu)|||_p \geq |||\lambda^p_G(\mu)|||_p$ is then a direct consequence of Theorem 3.1. □

REMARK 6.5. The map $a \mapsto \lambda^2_G(\delta_a)$ is the left regular representation of $G$ on $L^2(G)$. Theorem 6.1 is a version when $p \neq 2$ of the result of [3, Lemma 2, p. 606] and [4, Theorem 2, p. 3152]. The Hilbert space methods used to prove the version when $p = 2$ are not applicable when $p \neq 2$. Our proof requires another approach.

THEOREM 6.6. *Let $1 < p < \infty$ and $G$ be a locally compact group. Assume that $G_d$ is amenable. Then there is a linear contraction*

$$\sigma : CV_p(G) \to CV_p(G_d)$$

*such that, for all discrete measures $\mu$ on $G$,*

$$\sigma(\lambda^p_G(\mu)) = \lambda^p_{G_d}(\mu).$$

The proof of this theorem is based on Definition 6.2 and the following construction.

DEFINITION 6.7. Let $1 < p < \infty$, let $T \in CV_p(G)$, let $W$ be a relatively compact neighborhood of $e$ in $G$ and let $k, l \in C_{oo}(G_d)$. We define $\sigma_{W,k,l}(T)$ by

$$\begin{aligned}
&\langle \sigma_{W,k,l}(T)\varphi, \psi\rangle_{L^p(G_d),L^{p'}(G_d)} \\
&\quad = \sum_{x\in G} \langle T(\tau_p(T^p_W(k(_x\check{\varphi})))), \tau_{p'}(T^{p'}_W(l(_x\check{\psi})))\rangle_{L^p(G),L^{p'}(G)}
\end{aligned}$$

for all $\varphi \in L^p(G_d)$ and $\psi \in L^{p'}(G_d)$.

Let $\mathcal{W}$ denote the set of pairs $(W, r)$ where $W$ is a relatively compact neighborhood of $e$ in $G$, $r \in C_{oo}(G_d)$ such that $x_i W \cap x_j W = \emptyset$, for all $i \neq j$, $1 \leq i, j \leq n$, where $\{x_1, \ldots, x_n\} = \operatorname{supp}(r)$.

LEMMA 6.8. *Let $(W, r) \in \mathcal{W}$, $k = r^{1/p}$ and $l = r^{1/p'}$. Then $\sigma_{W,k,l}$ is a linear map of $CV_p(G)$ into $CV_p(G_d)$ with $|||\sigma_{W,k,l}(T)|||_p \leq |||T|||_p \, \|k\|_p \, \|l\|_{p'}$.*

LEMMA 6.9. *Let $\mu$ be a bounded measure of $G$ with finite support. Then there is $(W, r) \in \mathcal{W}$ such that, for all $\varphi, \psi \in C_{oo}(G_d)$,*

$$\langle \sigma_{W,k,l}(\lambda_G^p(\mu))\varphi, \psi\rangle_{L^p(G_d),L^{p'}(G_d)} = \overline{\mu^\star((\bar{k} \star \check{l})(\tilde{\varphi} \star \psi))},$$

*where $k = r^{1/p}$ and $l = r^{1/p'}$.*

PROOF OF THEOREM 6.6. For each $(W, r) \in \mathcal{W}$, $k = r^{1/p}$ and $l = r^{1/p'}$, we define

$$\Sigma_{W,k,l}(T, \varphi, \psi) = \langle \sigma_{W,k,l}(T)\varphi, \psi\rangle_{L^p(G_d),L^{p'}(G_d)},$$

where $T \in CV_p(G)$ and $\varphi, \psi \in C_{oo}(G_d)$. Here $\Sigma_{W,k,l}$ is a continuous form on $CV_p(G) \times L^p(G_d) \times L^{p'}(G_d)$, which is bilinear in the two first factors and conjugate linear on the third. Let $\mathcal{B}$ denote the set of these forms with the weak topology of duality with $CV_p(G) \times L^p(G_d) \times L^{p'}(G_d)$. By the Banach–Alaoglu theorem, $\mathcal{S} = \{F \in \mathcal{B} : |F(T, \varphi, \psi)| \le |||T|||_p \|\varphi\|_p \|\psi\|_{p'}\}$ is a compact subset of $\mathcal{B}$. For each $K$ finite subset of $G$, $\varepsilon > 0$ and $U$ neighborhood of $e$ in $G$, we define

$$\mathcal{A}_{K,\varepsilon,U} = \{\Sigma_{W,k,l} : (W, r) \in \mathcal{W},\ k = r^{1/p},\ l = r^{1/p'},\ r \ge 0,\ \|r\|_1 = 1, \\ \|{}_{x^{-1}}k - k\|_p < \varepsilon \quad \forall x \in K,\ W \subset U\}.$$

The $\mathcal{A}_{K,\varepsilon,U}$ are all nonempty, because $G_d$ is amenable. It easy to show that for all $n \in \mathbb{N}$, $K_1, \dots, K_n \subset G$ finite, $\varepsilon_1, \dots, \varepsilon_n > 0$ and $U_1, \dots, U_n$ neighborhood of $e$ on $G$, $\bigcap_{i=1}^n \mathcal{A}_{K_i,\varepsilon_i,U_i} \ne \emptyset$. However, $\mathcal{S}$ is compact, so there is

$$\Sigma \in \bigcap_{\substack{K \in G \text{ finite} \\ \varepsilon > 0 \\ U \text{ neighbor of } e}} \overline{\mathcal{A}_{K,\varepsilon,U}}.$$

For each $T \in CV_p(G)$, $\varphi \in L^p(G_d)$ and $\psi \in L^{p'}(G_d)$, we define

$$\Sigma(T, \varphi, \psi) = \langle \sigma(T)\varphi, \psi\rangle_{L^p(G_d),L^{p'}(G_d)}.$$ □

This extends Lust-Piquard's result [19, Theorem 4.1]. The techniques used for the proof are completely different and are not applicable to nonabelian groups. This problem was also treated by Lohoué in [17, 18] for special kinds of convolution operators, with strong use of structure theory of locally compact abelian groups.

REMARK 6.10. For $G$ amenable, the map defined in Theorem 6.6 could be considered as a substitute for the map of the $p$-multipliers of $\widehat{G}$ into the $p$-multipliers of the Bohr compactification of $\widehat{G}$.

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