Proofs of Lemmas of the Paper Design of a Distributed Quantized Luenberger Filter for Bounded Noise

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Proof of Lemma 2. Notice first that we can express the estimated state z^i_{t,l_f} as the average of the estimated states plus an error Y^i_{t,l_f}, i.e. $z^i_{t,l_f} = \sum_{j\in\mathcal{N}} \frac{1}{N} z^j_{t,l_f} + Y^i_{t,l_f}$, where Y^i_{t,l_f} is the component of Y_{t,l_f} corresponding to the node i. From the fact that the consensus algorithm preserves averages we have that $z^i_{t,l_f} = \sum_{j\in\mathcal{N}} \frac{1}{N} z^j_{t,0} + Y^i_{t,l_f}$. Then from the state dynamics and filter update equations (1) and (5), and the definitions of Φ^i, W^i_t and Γ^i_t we obtain equation (14) as follows

$$\begin{aligned}
e^i_{t+1,0} &= A(x_t - z^i_{t,l_f}) - L^i(C^i x_t + v^i_t - C^i z^i_{t,l_f}) + w_t \\
&= A(x_t - \sum_{j\in\mathcal{N}} \frac{1}{N} z^j_{t,0} - Y^i_{t,l_f}) \\
&\quad - L^i(C^i x_t + v^i_t - C^i \sum_{j\in\mathcal{N}} \frac{1}{N} z^j_{t,0} - C^i Y^i_{t,l_f}) \\
&\quad + w_t \\
&= \Phi^i(x_t - \sum_{j\in\mathcal{N}} \frac{1}{N} z^j_{t,0}) - \Gamma^i_t + W^i_t \\
&= \sum_{j\in\mathcal{N}} \frac{1}{N} \Phi^i e^j_{t,0} - \Gamma^i_t + W^i_t.
\end{aligned}$$

From the definitions of Φ, Γ_t and W_t we obtain directly equation (15)

$$\begin{aligned}
e_{t+1,0} &= \Phi e_{t,0} - \Gamma_t + W_t \\
&= \frac{1}{N} \operatorname{col}\left(\Phi^i\right) \mathbf{1}^T \otimes I_n e_{t,0} - \Gamma_t + W_t.
\end{aligned}$$

Since we can observe that $\operatorname{col}\left(\Phi^i\right)$ is equal to $\operatorname{diag}\left(\Phi^i\right) \mathbf{1}\otimes I_n$ the previous equation is equivalent to

$$e_{t+1,0} = \frac{1}{N} \operatorname{diag}\left(\Phi^i\right) \mathbf{1}\otimes I_n \mathbf{1}^T \otimes I_n e_{t,0} - \Gamma_t + W_t.$$

Using the former equation, the mixed-product property of the Kronecker product[1] and the definition of $e^{\text{avg}}_{t,0}$ we obtain equation (16) as follows

$$\begin{aligned}
e_{t+1,0} &= \operatorname{diag}\left(\Phi^i\right) \frac{1}{N} \left(\mathbf{1}\mathbf{1}^T\right) \otimes I_n e_{t,0} - \Gamma_t + W_t \\
&= \operatorname{diag}\left(\Phi^i\right) e^{\text{avg}}_{t,0} - \Gamma_t + W_t.
\end{aligned}$$

Finally, from the definition of $e^{\text{avg}}_{t+1,0}$ and equation (16) we have

$$\begin{aligned}
e^{\text{avg}}_{t+1,0} &= \frac{1}{N} \left(\mathbf{1}\mathbf{1}^T\right) \otimes I_n e_{t+1,0} \\
&= \frac{1}{N} \left(\mathbf{1}\mathbf{1}^T\right) \otimes I_n \left(\operatorname{diag}\left(\Phi^i\right) e^{\text{avg}}_{t,0}\right. \\
&\quad \left. - \Gamma_t + W_t\right).
\end{aligned}$$

Since $\mathbf{1}^T \otimes I_n \operatorname{diag}\left(\Phi^i\right)$ is equal to $\operatorname{row}\left(\Phi^i\right)$ and from the mixed-product property of the Kronecker product we have

$$\begin{aligned}
e^{\text{avg}}_{t+1,0} &= \frac{1}{N}\mathbf{1}\otimes I_n \operatorname{row}\left(\Phi^i\right) e^{\text{avg}}_{t,0} \\
&\quad + \frac{1}{N} \left(\mathbf{1}\mathbf{1}^T\right) \otimes I_n \left(W_t - \Gamma_t\right).
\end{aligned}$$

Noting that $\frac{1}{N}\left(\mathbf{1}\mathbf{1}^T\right) \otimes I_n e^{\text{avg}}_{t,0}$ is equal to $e^{\text{avg}}_{t,0}$ we have

$$\begin{aligned}
e^{\text{avg}}_{t+1,0} &= \frac{1}{N}\mathbf{1}\otimes I_n \operatorname{row}\left(\Phi^i\right) \frac{1}{N} \left(\mathbf{1}\mathbf{1}^T\right) \otimes I_n e^{\text{avg}}_{t,0} \\
&\quad + \frac{1}{N} \left(\mathbf{1}\mathbf{1}^T\right) \otimes I_n \left(W_t - \Gamma_t\right).
\end{aligned}$$

Using the mixed-product property and the fact that $\operatorname{row}\left(\Phi^i\right) \frac{1}{N}\mathbf{1}\otimes I_n = \frac{1}{N}\sum_{j\in\mathcal{N}} \Phi^j = A - LC$ the former equation is equivalent to

$$\begin{aligned}
e^{\text{avg}}_{t+1,0} &= \frac{1}{N}\mathbf{1}\otimes I_n \left(A - LC\right) \mathbf{1}^T \otimes I_n e^{\text{avg}}_{t,0} \\
&\quad + \frac{1}{N} \left(\mathbf{1}\mathbf{1}^T\right) \otimes I_n \left(W_t - \Gamma_t\right).
\end{aligned}$$

Again, using the mixed-product property we have that

$$\mathbf{1}\otimes I_n \left(A - LC\right) = I_N \otimes \left(A - LC\right) \mathbf{1}\otimes I_n.$$

And therefore it follows that

$$\begin{aligned}
e^{\text{avg}}_{t+1,0} &= I_N \otimes \left(A - LC\right) \frac{1}{N}\mathbf{1}\otimes I_n \mathbf{1}^T \otimes I_n e^{\text{avg}}_{t,0} \\
&\quad + \frac{1}{N} \left(\mathbf{1}\mathbf{1}^T\right) \otimes I_n \left(W_t - \Gamma_t\right).
\end{aligned}$$

And finally, from the former equation, the definition of $e^{\text{avg}}_{t,0}$ and the mixed-product property we obtain equation (17). □

Proof of Lemma 3. 1) Since it is given by assumption that for $t \le p \le 0$ we are under the conditions

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[1]Given four matrices M_1, M_2, M_3 and M_4 of proper size, the mixed-product property consists of the fact that $(M_1 \otimes M_2)(M_3 \otimes M_4) = (M_1 M_3) \otimes (M_2 M_4)$.

of Lemma 1, and that assumption A2 holds, then noting that $\|e^{\text{avg}}_{0,0}\| \leq \|e_{0,0}\|$ and that $\|e_{0,0}\| \leq \max\left(1, \frac{\bar{\Phi}}{\tilde{\beta}}\right)\|e_{0,0}\|$ applying equations (19) and (20) recursively we obtain

$$
\begin{aligned}
&\|e^{\text{avg}}_{p+1,0}\| \leq \tilde{\beta}\|e^{\text{avg}}_{p,0}\| + \bar{\Phi}\alpha^{l_f}\|e_{p,0}\| \\
&\quad + \bar{\Phi}\alpha^{l_f}k_6\tfrac{a\beta^p+b}{2^{n_b}} + \epsilon \\
&\quad \leq \bar{\beta}\left(\tilde{\beta}\|e^{\text{avg}}_{p-1,0}\| + \bar{\Phi}\alpha^{l_f}\|e_{p-1,0}\|\right. \\
&\quad \left. + \bar{\Phi}\alpha^{l_f}k_6\tfrac{a\beta^{p-1}+b}{2^{n_b}} + \epsilon\right) \\
&\quad + \bar{\Phi}\alpha^{l_f}k_6\tfrac{a\beta^p+b}{2^{n_b}} + \epsilon \\
&\quad = \bar{\beta}\left(\tilde{\beta}\|e^{\text{avg}}_{p-1,0}\| + \bar{\Phi}\alpha^{l_f}\|e_{p-1,0}\|\right) \\
&\quad + \textstyle\sum_{\tau=0}^{1}\bar{\beta}^{\tau}\left(\bar{\Phi}\alpha^{l_f}k_6\tfrac{a\beta^{p-\tau}+b}{2^{n_b}} + \epsilon\right),
\end{aligned}
$$

where $\bar{\beta}$ is defined in (21) and is strictly positive and smaller than 1 by assumption. Repeating this step p times we have

$$
\begin{aligned}
\|e^{\text{avg}}_{p+1,0}\| &\leq \bar{\beta}^{p+1}\|e_{0,0}\| \\
&+ \sum_{\tau=0}^{p}\bar{\beta}^{\tau}\left(\bar{\Phi}\alpha^{l_f}k_6\frac{a\beta^{p-\tau}+b}{2^{n_b}} + \epsilon\right) \\
&\leq \bar{\beta}^{p+1}\left[\|e_{0,0}\| + \alpha^{l_f}\bar{\Phi}k_6\frac{a}{2^{n_b}}\sum_{\tau=0}^{p}\bar{\beta}^{\tau-p-1}\beta^{p-\tau}\right] \\
&+ \epsilon\sum_{\tau=0}^{p}\bar{\beta}^{\tau} + \bar{\Phi}\alpha^{l_f}k_6\frac{b}{2^{n_b}}\sum_{\tau=0}^{p}\bar{\beta}^{\tau} \\
&\leq \beta^{p+1}\left[\|e_{0,0}\| + \alpha^{l_f}\bar{\Phi}k_6\frac{a}{2^{n_b}}\sum_{\tau=0}^{p}\frac{\bar{\beta}^{\tau}}{\beta^{\tau+1}}\right] \\
&+ \epsilon\sum_{\tau=0}^{p}\bar{\beta}^{\tau} + \bar{\Phi}\alpha^{l_f}k_6\frac{b}{2^{n_b}}\sum_{\tau=0}^{p}\bar{\beta}^{\tau}.
\end{aligned}
$$

Since $0 < \beta < 1$, by using the property of the geometric series, we get that the expression above is equal to

$$
\begin{aligned}
&\|e^{\text{avg}}_{p+1,0}\| \leq \\
&\leq \quad \beta^{p+1}\left[\|e_{0,0}\| + \frac{\bar{\Phi}\alpha^{l_f}k_6\left(1-\left(\frac{\bar{\beta}}{\beta}\right)^{p+1}\right)}{\beta\left(1-\frac{\bar{\beta}}{\beta}\right)}\frac{a}{2^{n_b}}\right] \\
&+ \quad \tfrac{\epsilon}{1-\bar{\beta}} + \tfrac{\bar{\Phi}\alpha^{l_f}k_6}{1-\bar{\beta}}\tfrac{b}{2^{n_b}} \\
&\leq \quad \beta^{p+1}\left[\|e_{0,0}\| + \tfrac{\bar{\Phi}\alpha^{l_f}k_6}{\beta-\bar{\beta}}\tfrac{a}{2^{n_b}}\right] \\
&+ \quad \tfrac{\epsilon}{1-\bar{\beta}} + \tfrac{\bar{\Phi}\alpha^{l_f}k_6}{1-\bar{\beta}}\tfrac{b}{2^{n_b}}.
\end{aligned}
$$

2) Similarly to the previous point, applying equations (19) and (20) recursively, and following the same steps as previously we have for $\|e_{p,0}\|$, for any p such that $t+1 \geq p \geq 0$.

$$
\begin{aligned}
\|e_{p,0}\| \quad &\leq \quad \max\left(1, \tfrac{\bar{\Phi}}{\tilde{\beta}}\right)\left(\beta^p\left[\|e_{0,0}\| + \tfrac{\bar{\Phi}\alpha^{l_f}k_6}{\beta-\bar{\beta}}\tfrac{a}{2^{n_b}}\right]\right. \\
&+ \quad \left.\tfrac{\epsilon}{1-\bar{\beta}} + \tfrac{\bar{\Phi}\alpha^{l_f}k_6}{1-\bar{\beta}}\tfrac{b}{2^{n_b}}\right).
\end{aligned}
$$

3) We have from (18) that

$$
\begin{aligned}
&\|Y_{p,0}\| \leq \|e_{p,0}\| \\
&\quad \leq \max\left(1, \frac{\bar{\Phi}}{\tilde{\beta}}\right)\left(\beta^p\left[\|e_{0,0}\| + c_8\frac{a}{2^{n_b}}\right]\right. \\
&\quad \left. + \frac{\epsilon}{1-\bar{\beta}} + d_8\frac{b}{2^{n_b}}\right), \forall t+1 \geq p \geq 0.
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
\|Y_{p,l}\| \quad &\leq \quad \alpha^l\left[\|Y_{p,0}\| + k_6\tfrac{a\beta^p+b}{2^{n_b}}\right] \\
&\leq \quad \alpha^l\left[\beta^p\left[\max\left(1, \tfrac{\bar{\Phi}}{\tilde{\beta}}\right)\|e_{0,0}\| + c_7\tfrac{a}{2^{n_b}}\right]\right. \\
&+ \quad \left.\tfrac{\max\left(1,\frac{\bar{\Phi}}{\tilde{\beta}}\right)\epsilon}{1-\tilde{\beta}} + d_7\tfrac{b}{2^{n_b}}\right], \\
&\forall \quad t \geq p \geq 0, l_f \geq l \geq 0.
\end{aligned}
$$

from Lemma 1.

4) Then we note that since $z_{p,l_f} = Y_{p,l_f} + z^{\text{avg}}_{p,l_f} = Y_{p,l_f} + z^{\text{avg}}_{p,0}$, from the fact that the consensus algorithm preserves averages, and $x_p = \frac{1}{N}\sum_{i\in\mathcal{N}} e^i_{p,0} + z^i_{p,0}$ we have

$$
\begin{aligned}
z^i_{p+1,0} &= Az^i_{p,l_f} + L^i\left(y^i_p - C^i z^i_{p,l_f}\right) \\
&= \quad \Phi^i z^i_{p,l_f} + L^i y^i_p \\
&= \quad \Phi^i z^i_{p,l_f} + L^i\left(C^i x_p + v^i_p\right) \\
&= \quad \Phi^i z^i_{p,l_f} + L^i C^i x_p + L^i v^i_p \\
&= \quad \Phi^i\left(Y^i_{p,l_f} + \tfrac{1}{N}\textstyle\sum_{j\in\mathcal{N}} z^j_{p,0}\right) \\
&+ \quad L^i C^i\left(\tfrac{1}{N}\textstyle\sum_{j\in\mathcal{N}} e^j_{p,0} + z^j_{p,0}\right) + L^i v^i_p \\
&= \quad \Phi^i Y^i_{p,l_f} + A\tfrac{1}{N}\textstyle\sum_{j\in\mathcal{N}} z^j_{p,0} \\
&+ \quad L^i C^i\tfrac{1}{N}\textstyle\sum_{j\in\mathcal{N}} e^j_{p,0} + L^i v^i_p.
\end{aligned}
$$

Therefore for the vector $z_{p+1,0}$ we have

$$
\begin{aligned}
z_{p+1,0} &= \text{diag}\left(\Phi^i\right)Y_{p,l_f} \\
&+ \quad I_N \otimes Az^{\text{avg}}_{p,0} \\
&+ \quad \text{diag}\left(L^i C^i\right)\tfrac{1}{N}\left(\mathbf{1}\mathbf{1}^T\right)\otimes I_n e_{p,0} \\
&+ \quad \text{col}\left(L^i v^i_p\right),
\end{aligned}
$$

and, noting that $\sum_{i\in\mathcal{N}}\left(L^i C^i\right) = NLC$, we have

$$
\begin{aligned}
z^{\text{avg}}_{p+1,0} &= \tfrac{1}{N}\left(\mathbf{1}\mathbf{1}^T\right)\otimes I_n\,\text{diag}\left(\Phi^i\right)Y_{t,l_f} \\
&+ \quad I_N \otimes Az^{\text{avg}}_{p,0} \\
&+ \quad I_N \otimes (LC)\tfrac{1}{N}\left(\mathbf{1}\mathbf{1}^T\right)\otimes I_n e_{p,0} \\
&+ \quad \tfrac{1}{N}\left(\mathbf{1}\mathbf{1}^T\right)\otimes I_n\,\text{col}\left(L^i v^i_p\right).
\end{aligned}
$$

For the vector $\bar{z}_{p+1,0}$ we have

$$
\begin{aligned}
\bar{z}_{p+1,0} &= I_N \otimes AQ_{p,l_f-1}\left(z_{p,l_f-1}\right) \\
&= \quad I_N \otimes A\left[Q_{p,L-1}\left(z_{p,l_f-1}\right) - z_{p,l_f-1}\right] \\
&+ \quad I_N \otimes Az_{pl_f-1} \\
&= \quad I_N \otimes A\left[Q_{p,l_f-1}\left(z_{p,l_f-1}\right) - z_{p,l_f-1}\right] \\
&+ \quad I_N \otimes AY_{p,l_f-1} \\
&+ \quad I_N \otimes Az^{\text{avg}}_{p,0},
\end{aligned}
$$

and finally

$$
\begin{aligned}
&\|\bar{z}_{p+1,0} - z^{\text{avg}}_{p+1,0}\| \leq \|A\|\tfrac{(a\beta^p+b)\alpha^{l_f-1}\sqrt{Nn}}{2^{n_b+1}} \\
&+ \quad \|A\|\|Y_{p,l_f-1}\| \\
&+ \quad \bar{\Phi}\|Y_{p,l_f}\| + \|LC\|\,\|e_{p,0}\| \\
&+ \quad \sqrt{N}\max_{j\in\mathcal{N}}\|L^j\|\epsilon^j_v \\
&\leq \quad c_5\beta^p\|e_{0,0}\| + c_6\beta^t\tfrac{a}{2^{n_b}} + d_5 + d_6\tfrac{b}{2^{n_b}}.
\end{aligned}
$$

5) Since $z_{p,l_f} = Y_{p,l_f} + z_{p,0}^{\text{avg}}$, which, subtracting both sides by $\mathbf{1} \otimes x_p$, is equivalent to $e_{p,l_f} = Y_{p,l_f} + e_{p,0}^{\text{avg}}$ we have for the norm of e_{p,l_f}

$$
\begin{aligned}
&\|e_{p,l_f}\| \leq \|Y_{p,l_f}\| + \|e_{p,0}^{\text{avg}}\| \\
&\leq \quad \beta^p \left[\left(1 + \alpha^{l_f} \max\left(1, \tfrac{\bar{\Phi}}{\tilde{\beta}}\right)\right) \|e_{0,0}\| \right. \\
&+ \quad \left. \left(c_8 + \alpha^{l_f} c_7\right) \tfrac{a}{2^{n_b}} \right] \\
&+ \quad \left(1 + \alpha^{l_f} \max\left(1, \tfrac{\bar{\Phi}}{\tilde{\beta}}\right)\right) \tfrac{\epsilon}{1-\tilde{\beta}} \\
&+ \quad \left(d_8 + \alpha^{l_f} d_7\right) \tfrac{b}{2^{n_b}}, \forall t+1 \geq p \geq 1,
\end{aligned}
$$

□

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