

# SPECTRAL METHODS FOR TEMPERED FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we first introduce fractional integral spaces, which possess some features: (i) when  $0 < \alpha < 1$ , functions in these spaces are not required to be zero on the boundary; (ii) the tempered fractional operators are equivalent to the Riemann-Liouville operator in the sense of the norm. Spectral Galerkin and Petrov-Galerkin methods for tempered fractional advection problems and tempered fractional diffusion problems can be developed as the classical spectral Galerkin and Petrov-Galerkin methods. Error analysis is provided and numerically confirmed for the tempered fractional advection and diffusion problems.

## 1. INTRODUCTION

Fractional calculus is a powerful mathematical tool to describe anomalous diffusion or dispersion phenomena, where a Lévy flight particle plume in the continuous time random walk (CTRW) model spreads at a rate inconsistent with the classical Brownian motion [6]. More specifically, the Lévy flight particle plume obeys a power-law waiting time distribution in the subdiffusion and the power-law jump length distribution in the superdiffusion, allowing the particle may exhibit arbitrarily large jumps or long waiting time [21]. However, this may not be realistic for the physical processes [5]; e.g., the biological particles moving in viscous cytoplasm, displaying trapped dynamical behavior, just move in bounded physical space and have a finite lifespan. Exponentially tempering the Lévy measure is a suitable way to ensure that the moments of Lévy distributions are finite. This way, we recover the spatially [5] or temporally tempered fractional Fokker-Planck equation [23]. Other applications for the tempered fractional derivatives and tempered differential equations can be found, for instance, in poroelasticity [14], finance [4], ground water hydrology [23], and geophysical flows [24].

A growing number of numerical methods for solving the tempered fractional differential equations (TFDEs), predominantly are based on finite difference methods, including high order methods (cf. [4, 10, 19, 22] and the references therein). In applications, high order methods require high regularity as well as several derivatives being zero at the boundary point (in the point to point sense) of the real solution to be effective. Although the techniques proposed in [7, 31] can remove the nonphysical boundary requirement and maintain high order of convergence, requirements of high regularity on the whole domain are still needed. However, since the fractional

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derivatives involve singular kernel, the solutions of FDEs are usually singular near the boundaries.

Since fractional operators are non-local, spectral method appears to be a natural choice. Eigenfunctions of a fractional Sturm-Liouville operator are derived in [29], and the use of a spectral collocation matrix is proposed in [30]; spectral approximation results in weighted Sobolev spaces involving fractional derivatives are derived in [9], including also rigorous convergence analysis. Collocation methods using fractional Birkhoff interpolation basis functions are proposed in [17]. Since all of previous work is based on the “generalized Jacobian functions (GJFs)”  $\{(1+x)^\beta J_n^{\alpha,\beta}(x)\}_{n=0}^N$ , where  $J_n^{\alpha,\beta}(x)$  is Jacobi polynomial, the real solutions must be zero at the boundary points. Furthermore, this kind of bases function only solves the simple fractional ordinary problem  ${}_a D_x^\alpha u(x) = f(x)$ . Spectral collocation methods for fractional problems using classical interpolation basis functions with Legendre-Gauss-Lobatto or Chebyshev-Gauss-Lobatto collocated points are proposed in [20] and [28]. Spectral accuracy still requires high regularity of the solution on the whole domain, moreover, the computational costs can be very large. Recently, fractional substantial equations on an infinite domain have been discussed in [16] by modifying Laguerre functions, but only the simplest case  ${}_a D_x^\alpha u(x) = f(x)$  has been addressed there. Here  ${}_a D_x^\alpha$  is the fractional substantial differential operator. To the knowledge of the authors, until now, there is hardly any spectral work has been done to solve the TFDEs.

In this paper, we first study the fractional integral spaces, in which functions are not required to be zero at the boundary point. Tempered fractional operators are equal to the standard Riemann-Liouville operators in the sense of the discussed integral spaces. Hence, we can derive the variational formulation for the TFDEs using these spaces, and numerically solve the TFDEs with GJFs in the weak sense, instead of in point to point sense or in the weighted space. This is one of the main differences between the spectral methods in this paper and that of previous works.

The rest of the paper is organized as follows. In Section 2, we describe the fractional integral spaces and study the properties of the tempered operators in these spaces. Then in Section 3, we derive the variational formulations and the spectral methods for the tempered advection and diffusion problems, respectively, and we develop their convergence in detail. Associated numerical experiments are present in Section 4 to support the theoretical results.

## 2. FUNCTIONAL SPACES FOR TEMPERED FRACTIONAL DIFFERENTIAL EQUATIONS

**2.1. Tempered fractional operators.** In this part, we review the definitions and some properties of tempered fractional calculus. Let  $\Omega = [a, b]$  be a finite interval on the real axis  $\mathbb{R}$ .

**Definition 2.1** (Tempered fractional integrals [5]). For  $\alpha > 0, \lambda \geq 0$ , the left and the right tempered fractional integral of order  $\alpha$  are, respectively, defined as

$$(2.1) \quad {}_a I_x^{\alpha,\lambda} u(x) := e^{-\lambda x} {}_a I_x^\alpha (e^{\lambda x} u(x)) = \frac{1}{\Gamma(\alpha)} \int_a^x e^{-\lambda(x-s)} (x-s)^{\alpha-1} u(s) ds,$$

and

$$(2.2) \quad {}_x I_b^{\alpha,\lambda} u(x) := e^{\lambda x} {}_x I_b^\alpha (e^{-\lambda x} u(x)) = \frac{1}{\Gamma(\alpha)} \int_x^b e^{\lambda(x-s)} (s-x)^{\alpha-1} u(s) ds,$$

where  $\Gamma$  presents the Euler gamma function, and  ${}_a I_x^\alpha$  denotes the left Riemann-Liouville fractional integral operator

$$(2.3) \quad {}_a I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} u(s) ds,$$

and  ${}_x I_b^\alpha$  denotes the right Riemann-Liouville fractional integral operator

$$(2.4) \quad {}_x I_b^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} u(s) ds.$$

The tempered fractional integrals (2.1) and (2.2) reduce to their corresponding Riemann-Liouville integrals if  $\lambda = 0$ . For convenience, in the following, the fractional integrals (2.1) and (2.2) are sometimes denoted as  ${}_a D_x^{-\alpha, \lambda} u(x)$  and  ${}_x D_b^{-\alpha, \lambda} u(x)$ .

**Definition 2.2** (Tempered fractional derivatives [5]). For  $n-1 \leq \alpha < n, n \in \mathbb{N}^+, \lambda \geq 0$ , the left and the right tempered fractional derivative are defined, respectively, as

$$(2.5) \quad {}_a D_x^{\alpha, \lambda} u(x) = e^{-\lambda x} {}_a D_x^\alpha (e^{\lambda x} u(x)) = \frac{e^{-\lambda x}}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{e^{\lambda s} u(s)}{(x-s)^{\alpha-n+1}} ds,$$

and

$$(2.6) \quad {}_x D_b^{\alpha, \lambda} u(x) = e^{\lambda x} {}_x D_b^\alpha (e^{-\lambda x} u(x)) = \frac{e^{\lambda x}}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b \frac{e^{-\lambda s} u(s)}{(s-x)^{\alpha-n+1}} ds,$$

where  ${}_a D_x^\alpha$  and  ${}_x D_b^\alpha$  are, respectively, the left and the right Riemann-Liouville fractional derivative operators [25]

$$(2.7) \quad {}_a D_x^\alpha u(x) = D^n {}_a I_x^{n-\alpha} u(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{u(s)}{(x-s)^{\alpha-n+1}} ds,$$

and

$$(2.8) \quad {}_x D_b^\alpha u(x) = (-D)^n {}_x I_b^{n-\alpha} u(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b \frac{u(s)}{(s-x)^{\alpha-n+1}} ds.$$

## 2.2. Fractional integral spaces.

2.2.1. *Definitions.* We denote by  $L_p(a, b)$  ( $1 \leq p < \infty$ ) the set of Lebesgue measurable functions  $f$  on  $\Omega = [a, b]$  for which  $\|f\|_{L_p} < \infty$ , where  $\|f\|_p = \left( \int_a^b |f(x)|^p dx \right)^{1/p}$ ,  $1 \leq p < \infty$ .

Let  $AC[a, b]$  be the space of functions  $f$  which are absolutely continuous on  $[a, b]$ . It is known ([18], p. 338) that  $AC[a, b]$  coincides with the space of primitives of Lebesgue summable functions:

$$(2.9) \quad f(x) \in AC[a, b] \Leftrightarrow f(x) = c + \int_a^x \phi(x) dx, \quad \phi(x) \in L_1(a, b),$$

where  $\phi(x) = f'(x)$ , and  $c = f(a)$ .

Denote by  $AC^n[a, b]$  the space of functions  $f(x)$  which have continuous derivatives up to order  $n-1$  on  $[a, b]$  such that  $f^{(n-1)}(x) \in AC[a, b]$ :

$$(2.10) \quad AC^n[a, b] = \{f : D^{n-1} f(x) \in AC[a, b]\}.$$

In particular,  $AC^1[a, b] = AC[a, b]$ .

In this paper, we focus on the fractional integral space of functions  ${}_a I_x^\alpha(L_p)$  and  ${}_x I_b^\alpha(L_p)$ , for  $\alpha > 0$  and  $1 \leq p < \infty$ , defined by ([26], Definition 2.3)

$$(2.11) \quad {}_a I_x^\alpha(L_p) := \{f : f(x) = {}_a I_x^\alpha \varphi(x), \varphi(x) \in L_p(a, b)\},$$

$$(2.12) \quad {}_x I_b^\alpha(L_p) := \{f : f(x) = {}_x I_b^\alpha \varphi(x), \varphi(x) \in L_p(a, b)\},$$

respectively.

Let us consider some properties of the defined fractional spaces. We only discuss  ${}_a I_x^\alpha(L_p)$ ; similar results can be derived for  ${}_x I_b^\alpha(L_p)$ .

**Lemma 2.3.** *If  $f(x) \in {}_a I_x^\alpha(L_p)$ , then there exists a unique  $\varphi(x) \in L_p(a, b)$ , such that  $f(x) = {}_a I_x^\alpha \varphi(x)$ .*

*Proof.* If there are two functions  $\varphi_1(x)$  and  $\varphi_2(x)$  in  $L_p(a, b)$ , such that

$$(2.13) \quad f(x) = {}_a I_x^\alpha \varphi_1(x) = {}_a I_x^\alpha \varphi_2(x),$$

then

$$(2.14) \quad {}_a I_x^\alpha (\varphi_1(x) - \varphi_2(x)) \equiv 0, \quad a \leq x \leq b.$$

Assume that  $\nu(x) := \varphi_1(x) - \varphi_2(x) \neq 0$  in the sense of an  $L_p$  norm. Then since  $C^\infty([a, b])$  is dense in  $L_p(a, b)$  ([13], p. 252), there exist  $\{\nu_n(x)\} \in C^\infty(a, b)$ ,  $(a_1, b_1) \subseteq (a, b)$ , such that  $\nu_n(x) \rightarrow \nu(x)$ ,  $n \rightarrow \infty$ . Furthermore, there is  $N > 0$ , and  $n > N$ ,  $\nu_n(x) = 0$  in  $(a, a_1)$ , while  $\nu_n(x) \neq 0$  in  $(a_1, b_1)$ , say  $\nu_n(x) > 0$ . Therefore, by the definition of the Riemann-Liouville fractional integral operator (2.3), we have

$$(2.15) \quad {}_a I_{b_1}^\alpha \nu_n(x) = {}_{a_1} I_{b_1}^\alpha \nu_n(x) > 0,$$

which contradicts to (2.14).  $\square$

The characterization of the space  ${}_a I_x^\alpha(L_1)$  is given by the following lemma.

**Lemma 2.4** ([26], Theorem 2.3). *In order that  $f(x) \in {}_a I_x^\alpha(L_1)$ ,  $n - 1 \leq \alpha < n$ , it is necessary and sufficient that  ${}_a I_x^{n-\alpha} f(x) \in AC^n[a, b]$ , and that*

$$(2.16) \quad {}_a D_x^{\alpha-k} f(x)|_{x=a} = 0, \quad k = 1, 2, \dots, n.$$

It is well known that [25] the equality  ${}_a D_x^\alpha {}_a I_x^\alpha f(x) = f(x)$  is valid for any summable function  $f(x)$ ; while for  $f(x) \in L_1(a, b)$  and  ${}_a D_x^\alpha f(x) \in L_1(a, b)$ , in stead of  $f(x) \in {}_a I_x^\alpha[L_1(a, b)]$ , we have

$$(2.17) \quad {}_a I_x^\alpha {}_a D_x^\alpha f(x) = f(x) - \sum_{k=1}^n \frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} {}_a D_x^{\alpha-k} f(x)|_{x=a}.$$

From Lemma 2.4, it is easy to see that the following Corollary holds.

**Corollary 2.5.** *If  $f(x) \in {}_a I_x^\alpha[L_1(a, b)]$ , then  ${}_a I_x^\alpha {}_a D_x^\alpha f(x) = f(x)$ .*

Therefore, we can see that:

- (i) By the definitions of  $AC[a, b]$  as well as  $AC^n[a, b]$  in (2.9)-(2.10), we have  ${}_a D_x^\alpha f(x) \in L_1(a, b)$  from  ${}_a I_x^{n-\alpha} f(x) \in AC^n[a, b]$ . So  ${}_a I_x^{n-\alpha} f(x) \in AC^n[a, b]$  or “ ${}_a D_x^\alpha f(x)$  exists almost everywhere and is summable” is *necessary but not sufficient* for  $f(x) \in {}_a I_x^\alpha(L_1)$ .

- (ii) It is clear [25] that for  $k = 1, 2, \dots, n$ ,  ${}_a I_x^{n-\alpha}(x-a)^{\alpha-k} = \frac{\Gamma(\alpha-k+1)}{\Gamma(n-k+1)}(x-a)^{n-k} \in AC^n[a, b]$ , and  ${}_a D_x^{\alpha-k_1}(x-a)^{\alpha-k}|_{x=a} = 0$  for  $k_1 \neq k$ , but  ${}_a D_x^{\alpha-k}(x-a)^{\alpha-k} \equiv \Gamma(\alpha-k+1) \neq 0$ . Thus, by Lemma 2.4,  $(x-a)^{\alpha-k} \notin {}_a I_x^\alpha(L_1)$ . In other words, some functions, which may destroy the general of the regularity principle characterization, are excluded from the space  ${}_a I_x^\alpha(L_1)$ .

*Remark 2.6.* (i) We prove in Appendix A that if  $f(x) \in {}_a I_x^\alpha[L_1(a, b)]$ , then the condition (2.16) is equivalent to

$$(2.18) \quad D^j f(x)|_{x=a} = 0, \quad j = 0, 1, \dots, n-2, \quad (\text{if } n \geq 2); \quad \text{and } {}_a I_x^{n-\alpha} f(x)|_{x=a} = 0.$$

In particular, when  $0 < \alpha < 1$ ,  $f(x) \in {}_a I_x^\alpha[L_1(a, b)]$  **does not need to be zero** at the left boundary. In fact,  $f(a)$  can be a constant or even infinite. Let's see an example: assume that  $0 < \alpha \leq \alpha' < 1$ ,  $f(x) = (x-a)^{\alpha-\alpha'}$ , then

$$(2.19) \quad {}_a I_x^{1-\alpha} f(x) = \frac{\Gamma(\alpha-\alpha'+1)}{\Gamma(2-\alpha')} (x-a)^{1-\alpha'} \in AC[a, b],$$

because, by (2.9),

$$(2.20) \quad D({}_a I_x^{1-\alpha} f(x)) = \frac{\Gamma(\alpha-\alpha'+1)}{\Gamma(1-\alpha')} (x-a)^{-\alpha'} \in L_1(a, b).$$

Also,  ${}_a I_x^{1-\alpha} f(x)|_{x=a} = 0$ . Thus, by Lemma 2.4, we have  $f(x) \in {}_a I_x^\alpha[L_1(a, b)]$ . While  $f(x) \neq 0$ , when  $x \rightarrow a$ .

- (ii) It should be clarified that Corollary 2.5 is different from

$$(2.21) \quad D^n {}_a I_x^{n-\alpha} f(x) = {}_a I_x^{n-\alpha} D^n f(x) = {}_a D_x^\alpha f(x).$$

We know that [25] when  $f(x) \in AC^n[a, b]$ , we have

$$(2.22) \quad {}_a I_x^{n-\alpha} D^n f(x) = {}_a D_x^\alpha f(x) - \sum_{k=0}^{n-1} \frac{D^k f(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha}.$$

We also show in Appendix A that if  $f(x) \in AC^n[a, b]$ , and  $D^j f(a) = 0$ ,  $j = 0, 1, \dots, n-2$ , then  $f(x) \in {}_a I_x^\alpha[L_1(a, b)]$ . So, from (2.22) and (2.18), we get

$$(2.23) \quad {}_a I_x^{n-\alpha} D^n f(x) = {}_a D_x^\alpha f(x) - \frac{D^{n-1} f(a)}{\Gamma(n-\alpha)} (x-a)^{n-1-\alpha},$$

but DO NOT recover (2.21).

Since  $L_p(a, b)$  can be embedded into  $L_q(a, b)$  for  $1 \leq q < p < \infty$  [26], we have

$$(2.24) \quad {}_a I_x^\alpha[L_p(a, b)] \subseteq {}_a I_x^\alpha[L_q(a, b)].$$

By Lemma 2.3 and Corollary 2.5, we obtain that

**Lemma 2.7.**

$$(2.25) \quad f(x) \in {}_a I_x^\alpha[L_p(a, b)] \Leftrightarrow \exists \varphi(x) \in L_p(a, b), \text{ s.t. } f(x) = {}_a I_x^\alpha \varphi(x),$$

where  $\varphi(x) = {}_a D_x^\alpha f(x)$ .

It follows that  ${}_a I_x^\alpha \varphi(x) \equiv 0$ ,  $\varphi(x) \in L_p(a, b)$ , only in the case  $\varphi(x) \equiv 0$ . Thus we can introduce the norm in  ${}_a I_x^\alpha [L_p(a, b)]$  by

$$(2.26) \quad \|f(x)\|_{{}_a I_x^\alpha [L_p(a, b)]} = \|{}_a D_x^\alpha f(x)\|_p.$$

It is easy to see that the space  ${}_a I_x^\alpha [L_p(a, b)]$  with the norm (2.26) is a Banach space as it is isometric to  $L_p(a, b)$ .

Now, we can show another characterization of the fractional integral space.

**Lemma 2.8** ([26], p.208 and p.337). *When  $0 < \alpha < 1/p$ , and  $1 < p < \infty$ , then*

$$(2.27) \quad H^{\alpha, p}[a, b] = I^\alpha [L_p(a, b)] := {}_a I_x^\alpha [L_p(a, b)] = {}_x I_b^\alpha [L_p(a, b)],$$

*up to the equivalence of norms. When  $1/p < \alpha < 1/p + 1$ , then*

$$(2.28) \quad H_0^{\alpha, p}[a, b] = {}_a I_x^\alpha [L_p(a, b)];$$

*where*

$$(2.29) \quad H_0^{\alpha, p}[a, b] = \{f : f(x) \in H^{\alpha, p}[a, b], \text{ and } f(a) = 0\},$$

$$(2.30) \quad H^{\alpha, p}[a, b] = \left\{f : \exists g(x) \in H^{\alpha, p}(\mathbb{R}), \text{ s.t. } g(x)|_{[a, b]} = f(x)\right\},$$

*with the norm  $\|f(x)\|_{H^{\alpha, p}[a, b]} = \inf \|g(x)\|_{H^{\alpha, p}(\mathbb{R})}$ .*

**2.2.2. Properties.** Let us now discuss properties of the fractional integral spaces introduced, to recover some useful results related to the tempered calculus.

**Lemma 2.9** (Theorem 2.6, Theorem 3.5, Theorem 3.6 in [26]). (i) *The fractional integration operator  ${}_a I_x^\alpha$  is bounded in  $L_p(a, b)$  ( $p \geq 1$ ):*

$$(2.31) \quad \|{}_a I_x^\alpha \varphi(x)\|_p \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|\varphi(x)\|_p.$$

(ii) *If  $0 < \alpha < 1$  and  $1 < p < 1/\alpha$ , then the fractional integration operator  ${}_a I_x^\alpha$  is bounded from  $L_p(a, b)$  into  $L_q(a, b)$ , where  $q = \frac{p}{1-\alpha p}$ .*

(iii) *If  $p > 1/\alpha$ , the fractional integral operator  ${}_a I_x^\alpha$  is bounded from  $L_p(a, b)$  into  $C^{n_0, \alpha-1/p-n_0}[a, b]$  if  $\alpha-1/p \notin \mathbb{N}^+$ , and  ${}_a I_x^\alpha \varphi(x) = o((x-a)^{\alpha-1/p})$  as  $x \rightarrow a$ . where  $n_0 = \lfloor \alpha - 1/p \rfloor$ , and  $C^{m, \gamma}[a, b]$  is a Hölder space defined by*

$$(2.32) \quad \begin{aligned} & C^{m, \gamma}[a, b] \\ &= \left\{f : f(x) \in C^m[a, b] \text{ and } \exists A > 0 \text{ s.t. } \frac{|D^m f(x) - D^m f(y)|}{|x-y|^\gamma} \leq A \forall x, y \in (a, b)\right\}. \end{aligned}$$

(iv) *If  $0 < 1/p < \alpha < 1 + 1/p$ , the fractional integral operator  ${}_a I_x^\alpha$  is bounded from  $L_p(a, b)$  into  $C^{n_0, \alpha-1/p-n_0}[a, b]$ , where  $n_0 = \lfloor \alpha - 1/p \rfloor$ ,  $C^{m, \gamma}[a, b]$  is defined by*

$$(2.33) \quad C^{m, \gamma}[a, b] = \left\{f : f(x) \in C^m[a, b] \text{ and } \frac{|D^m f(x) - D^m f(y)|}{|x-y|^\gamma} \rightarrow 0 \text{ as } x \rightarrow y\right\}.$$

**Remark 2.10.** Inequality (2.31) demonstrates the rationality of the norm (2.26), i.e.,  $\|{}_a D_x^\alpha f(x)\|_p$  is indeed a norm rather than a semi-norm if  $f(x) \in {}_a I_x^\alpha [L_p(a, b)]$ , although  $f(x)$  might not be zero on the left boundary.

**Lemma 2.11.** *If  $\alpha_1 > \alpha_2 \geq 0$ , then  ${}_a I_x^{\alpha_1} [L_p(a, b)] \subseteq {}_a I_x^{\alpha_2} [L_p(a, b)]$ .*

*Proof.* Firstly, if  $f(x) \in {}_a I_x^{\alpha_1}[L_p(a, b)]$ , then there is  $\varphi(x) \in L_p(a, b)$ , such that

$$(2.34) \quad f(x) = {}_a I_x^{\alpha_1} \varphi(x) = {}_a I_x^{\alpha_2} {}_a I_x^{\alpha_1 - \alpha_2} \varphi(x).$$

From Lemma 2.9 and  $\varphi(x) \in L_p(a, b)$ , we have  ${}_a I_x^{\alpha_1 - \alpha_2} \varphi(x) \in L_p(a, b)$ . Thus,  $f(x) \in {}_a I_x^{\alpha_2}[L_p(a, b)]$ .

Secondly, by (2.26) and (2.31), we have

$$(2.35) \quad \|f(x)\|_{{}_a I_x^{\alpha_2}[L_p(a, b)]} = \|{}_a I_x^{\alpha_1 - \alpha_2} \varphi(x)\|_p \leq C \cdot \|\varphi(x)\|_p = C \cdot \|f(x)\|_{{}_a I_x^{\alpha_1}[L_p(a, b)]}.$$

□

Next, we show that the tempered fractional operation is equivalent to fractional operation in the space  ${}_a I_x^\alpha[L_p(a, b)]$ .

**Lemma 2.12** (Lemma 10.3 in [26]). *Let  $\alpha > 0$ ,  $\sigma \in \mathbb{R}$ , and  $1 \leq p < \infty$ . Then*

$$(2.36) \quad e^{-\sigma x} {}_a I_x^\alpha e^{\sigma x} u(x) \in {}_a I_x^\alpha[L_p(a, b)] \Leftrightarrow u(x) \in L_p(a, b);$$

$$(2.37) \quad e^{-\sigma x} {}_a D_x^\alpha e^{\sigma x} u(x) \in L_p(a, b) \Leftrightarrow u(x) \in {}_a I_x^\alpha[L_p(a, b)].$$

**Corollary 2.13.** Let  $\alpha > 0$ ,  $\sigma \in \mathbb{R}$ , and  $1 \leq p < \infty$ . Then

$$(2.38) \quad u(x) \in {}_a I_x^\alpha[L_p(a, b)]$$

if and only if

$$(2.39) \quad e^{\sigma x} u(x) \in {}_a I_x^\alpha[L_p(a, b)],$$

and there are positive constants  $C_1$  and  $C_2$ , which depend on  $\alpha$ ,  $\sigma$ , and  $a$ ,  $b$ , such that

$$(2.40) \quad C_1 \|u(x)\|_{{}_a I_x^\alpha[L_p(a, b)]} \leq \|e^{\sigma x} u(x)\|_{{}_a I_x^\alpha[L_p(a, b)]} \leq C_2 \|u(x)\|_{{}_a I_x^\alpha[L_p(a, b)]}.$$

*Proof.* If  $u(x) \in {}_a I_x^\alpha[L_p(a, b)]$ , then there exists  $\nu(x) \in L_p(a, b)$ , such that

$$u(x) = {}_a I_x^\alpha \nu(x) = {}_a I_x^\alpha e^{-\sigma x} e^{\sigma x} \nu(x) := {}_a I_x^\alpha e^{-\sigma x} \tilde{\nu}(x),$$

where  $\tilde{\nu}(x) = e^{\sigma x} \nu(x) \in L_p(a, b)$ . Therefore,  $e^{\sigma x} u(x) = e^{\sigma x} {}_a I_x^\alpha e^{-\sigma x} \tilde{\nu}(x)$ . Then, by Lemma 2.12, we have  $e^{\sigma x} u(x) \in {}_a I_x^\alpha[L_p(a, b)]$ . Similarly, we have  $e^{-\sigma x} u(x) \in {}_a I_x^\alpha[L_p(a, b)]$ .

If  $e^{\sigma x} u(x) \in {}_a I_x^\alpha[L_p(a, b)]$ , then by the above analysis, we have

$$e^{-\sigma x} (e^{\sigma x} u(x)) = u(x) \in {}_a I_x^\alpha[L_p(a, b)].$$

Finally, by Eqs. (2.26), (2.31), and Lemma 2.12, we have

$$\|e^{\sigma x} u(x)\|_{{}_a I_x^\alpha[L_p(a, b)]} = \|e^{\sigma x} {}_a I_x^\alpha e^{-\sigma x} \tilde{\nu}(x)\|_{{}_a I_x^\alpha[L_p(a, b)]} \leq C \|\tilde{\nu}(x)\|_p \leq C_2 \|u(x)\|_{{}_a I_x^\alpha[L_p(a, b)]}.$$

The lower bound

$$C_1 \|u(x)\|_{{}_a I_x^\alpha[L_p(a, b)]} \leq \|e^{\sigma x} u(x)\|_{{}_a I_x^\alpha[L_p(a, b)]}$$

can be obtained in a similar way. □

From Corollary 2.13, Lemma 2.4, and Lemma A.1, the following corollary follows.

**Corollary 2.14.** Let  $n - 1 \leq \alpha < n$ , and  $\sigma \in \mathbb{R}$ . If  $u(x) \in {}_a I_x^\alpha [L_p(a, b)]$ , then  ${}_a I_x^{n-\alpha} (e^{\sigma x} u(x)) \in AC^n[a, b]$ , and

$$(2.41) \quad {}_a D_x^{\alpha-k} (e^{\sigma x} u(x)) \Big|_{x=a} = 0, \quad k = 1, 2, \dots, n.$$

Also,

$$(2.42) \quad D^j (e^{\sigma x} u(x)) \Big|_{x=a} = 0, \quad j = 0, 1, \dots, n - 2.$$

**Lemma 2.15.** Let  $\alpha_1 > 0$ ,  $\alpha_2 > 0$ ,  $\alpha_1 + \alpha_2 = \alpha$ , and  $\lambda > 0$ . If  $u(x) \in {}_a I_x^\alpha [L_p(a, b)]$ , then

$$(2.43) \quad {}_a D_x^{\alpha, \lambda} u(x) = {}_a D_x^{\alpha_1, \lambda} {}_a D_x^{\alpha_2, \lambda} u(x),$$

and

$$(2.44) \quad {}_a D_x^{\alpha_2, \lambda} u(x) \in {}_a I_x^{\alpha_1} [L_p(a, b)].$$

*Proof.* If  $u(x) \in {}_a I_x^\alpha [L_p(a, b)]$ , then from Corollary 2.13, we have  $e^{\lambda x} u(x) \in {}_a I_x^\alpha [L_p(a, b)]$ . From Lemma 2.11, there is  $e^{\lambda x} u(x) \in {}_a I_x^{\alpha_2} [L_p(a, b)]$ . So, by Corollary 2.14, we have

$${}_a D_x^{\alpha_2-k} (e^{\lambda x} u(x)) \Big|_{x=a} = 0, \quad k = 1, 2, \dots, \lceil \alpha_2 \rceil.$$

Thus ([25], p.74),

$$\begin{aligned} & {}_a D_x^{\alpha_1, \lambda} {}_a D_x^{\alpha_2, \lambda} u(x) \\ &= e^{-\lambda x} {}_a D_x^{\alpha_1} {}_a D_x^{\alpha_2} e^{\lambda x} u(x) \\ &= e^{-\lambda x} {}_a D_x^\alpha (e^{\lambda x} u(x)) - e^{-\lambda x} \sum_{j=1}^{\lceil \alpha_2 \rceil} \frac{(x-a)^{-\alpha_1-j}}{\Gamma(1-\alpha_1-j)} \cdot {}_a D_x^{\alpha_2-j} (e^{\lambda x} u(x)) \Big|_{x=a} \\ &= {}_a D_x^{\alpha, \lambda} u(x). \end{aligned}$$

Since  $e^{\lambda x} u(x) \in {}_a I_x^\alpha [L_p(a, b)]$ , there exists  $\varphi(x) \in L_p(x)$ , such that  $e^{\lambda x} u(x) = {}_a I_x^\alpha \varphi(x)$ , and

$${}_a D_x^{\alpha_2, \lambda} u(x) = e^{-\lambda x} {}_a D_x^{\alpha_2} e^{\lambda x} u(x) = e^{-\lambda x} {}_a D_x^{\alpha_2} {}_a I_x^\alpha \varphi(x) = e^{-\lambda x} {}_a I_x^{\alpha_1} \varphi(x).$$

Thus, we obtain  ${}_a D_x^{\alpha_2, \lambda} u(x) \in {}_a I_x^{\alpha_1} [L_p(a, b)]$  from Corollary 2.13.  $\square$

In the following, we mainly focus on specific cases  ${}_a I_x^\alpha [L_p(a, b)]$  and  ${}_x I_b^\alpha [L_p(a, b)]$  when  $p = 2$ .

**Lemma 2.16.** Let  $\alpha > 0$ ,  $\lambda > 0$ . If  $u(x) \in L_2(a, b)$ ,  $v(x) \in L_2(a, b)$ , then

$$(2.45) \quad ({}_a I_x^{\alpha, \lambda} u(x), v(x)) = (u(x), {}_x I_b^{\alpha, \lambda} v(x)).$$

*Proof.* By Definition 2.1 and Property A.2 of [12], (2.45) can be easily obtained.  $\square$

**Lemma 2.17.** Let  $n - 1 \leq \alpha < n$ ,  $\lambda > 0$ . If  $u(x) \in {}_a I_x^{\alpha, \lambda} [L_2(a, b)]$ ,  $v(x) \in {}_x I_b^{\alpha, \lambda} [L_2(a, b)]$ , then

$$(2.46) \quad ({}_a D_x^{\alpha, \lambda} u(x), v(x)) = (u(x), {}_x D_b^{\alpha, \lambda} v(x)).$$

*Proof.* If  $u(x) \in {}_a I_x^{\alpha, \lambda} [L_2(a, b)]$ ,  $v(x) \in {}_x I_b^{\alpha, \lambda} [L_2(a, b)]$ , then from Corollary 2.13, Lemmas 2.7 and 2.9, we have

$$u(x) \in L_2(a, b), \quad v(x) \in L_2(a, b), \quad \text{and} \quad {}_a D_x^{\alpha, \lambda} u(x) \in L_2(a, b), \quad {}_x D_b^{\alpha, \lambda} v(x) \in L_2(a, b).$$



From Corollary 2.5 and Lemma 2.16, we can easily get

$$({}_a D_x^{\alpha, \lambda} u(x), v(x)) = ({}_a D_x^{\alpha, \lambda} u(x), {}_x I_b^{\alpha, \lambda} {}_x D_b^{\alpha, \lambda} v(x)) = (u(x), {}_x D_b^{\alpha, \lambda} v(x)).$$

□

For any function  $h(x)$ ,  $x \in [a, b]$ , by means of zero extension, it can also be viewed as a function defined on  $\mathbb{R}$ . So, it can be noticed that if a function  $g(x) \in L_p(a, b)$ , then  $\widehat{g}(\omega) \in L_p(\mathbb{R})$ , where  $\widehat{g}(\omega) = \mathcal{F}(g)(\omega)$ , where  $\mathcal{F}[g(x)](\omega) = \int_{\mathbb{R}} e^{i\omega x} g(x) dx$  is the Fourier transform operators. For the tempered integration operators, it can be proved as in [8] that

$$(2.47) \quad \mathcal{F}[{}_a D_x^{-\alpha, \lambda} g(x)](\omega) = (\lambda - i\omega)^{-\alpha} \widehat{g}(\omega),$$

and

$$(2.48) \quad \mathcal{F}[{}_x D_b^{-\alpha, \lambda} g(x)](\omega) = (\lambda + i\omega)^{-\alpha} \widehat{g}(\omega).$$

But unfortunately, similar statement for the tempered fractional derivative

$$(2.49) \quad \mathcal{F}[{}_a D_x^{\alpha, \lambda} g(x)](\omega) = \mathcal{F}[{}_{-\infty} D_x^{\alpha, \lambda} g(x)](\omega) = (\lambda - i\omega)^{\alpha} \widehat{g}(\omega)$$

is NOT true, even when  $g(x) \in I^{\alpha}[L_p(a, b)]$ , where  $0 < \alpha < 1/p$ , and  $1 < p < \infty$ . However, we can still prove the following result, similar to Lemma 2.4 in [12].

**Lemma 2.18.** *Let  $0 < \alpha < \frac{1}{2}$ , and  $\lambda \geq 0$ . If  $f(x) \in L_p(a, b)$ ,  $p = \frac{2}{1+2\alpha}$ , then*

$$(2.50) \quad \left( {}_a I_x^{\alpha, \lambda} f(x), {}_x I_b^{\alpha, \lambda} f(x) \right) \geq \cos(\pi\alpha) \cdot \| {}_a I_x^{\alpha, \lambda} f(x) \|_2^2 = \cos(\pi\alpha) \cdot \| {}_x I_b^{\alpha, \lambda} f(x) \|_2^2.$$

*Proof.* Firstly, since  $1 < p = \frac{2}{1+2\alpha} < \frac{1}{\alpha}$ , from (ii) of Lemma 2.9, we can see that  ${}_a I_x^{\alpha, \lambda} f(x) \in L_2(a, b)$ . Also  ${}_x I_b^{\alpha, \lambda} f(x) \in L_2(a, b)$ .

By noticing that

$$(2.51) \quad (\lambda + i\omega)^{-\alpha} = |\lambda^2 + \omega^2|^{-\alpha/2} e^{-i\theta\alpha}, \text{ and } (\lambda - i\omega)^{-\alpha} = |\lambda^2 + \omega^2|^{-\alpha/2} e^{i\theta\alpha},$$

where  $\tan \theta = \frac{\omega}{\lambda}$ , we can get

$$(2.52) \quad \overline{(\lambda + i\omega)^{-\alpha}} = \overline{(\lambda - i\omega)^{-\alpha}} \cdot e^{2i\theta\alpha}, \text{ and } |\theta| \leq \frac{\pi}{2}.$$

Therefore, from Eqs. (2.47) and (2.48), we have

$$\begin{aligned}
& \left( {}_a I_x^{\alpha, \lambda} f(x), {}_x I_b^{\alpha, \lambda} f(x) \right) \\
&= \int_{-\infty}^{\infty} (\lambda - i\omega)^{-\alpha} \widehat{f}(\omega) \cdot \overline{(\lambda + i\omega)^{-\alpha} \widehat{f}(\omega)} d\omega \\
&= \int_{-\infty}^{\infty} (\lambda - i\omega)^{-\alpha} \widehat{f}(\omega) \cdot e^{2i\theta\alpha} (\lambda - i\omega)^{-\alpha} \widehat{f}(\omega) d\omega \\
&= \int_{-\infty}^{\infty} \cos(2\theta\alpha) \mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (\omega) \overline{\mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (\omega)} d\omega \\
&\quad + i \int_0^{\infty} \sin(2\theta\alpha) \left[ \mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (\omega) \overline{\mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (\omega)} d\omega \right. \\
(2.53) \quad &\quad \left. - \mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (-\omega) \overline{\mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (-\omega)} d\omega \right] d\omega \\
&= \int_{-\infty}^{\infty} \cos(2\theta\alpha) \mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (\omega) \overline{\mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (\omega)} d\omega \\
&\quad + i \int_0^{\infty} \sin(2\theta\alpha) \left[ \mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (\omega) \overline{\mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (-\omega)} d\omega \right. \\
&\quad \left. - \mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (-\omega) \overline{\mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (\omega)} d\omega \right] d\omega \\
&= \int_{-\infty}^{\infty} \cos(2\theta\alpha) \mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (\omega) \overline{\mathcal{F} \left( {}_a I_x^{\alpha, \lambda} f(x) \right) (\omega)} d\omega.
\end{aligned}$$

Since

$$(2.54) \quad |2\theta\alpha| \leq \alpha\pi < \frac{\pi}{2}, \text{ and } \cos(2\theta\alpha) \geq \cos(\alpha\pi) > 0,$$

we have

$$(2.55) \quad \left( {}_a I_x^{\alpha, \lambda} f(x), {}_x I_b^{\alpha, \lambda} f(x) \right) \geq \cos(\alpha\pi) \|{}_a I_x^{\alpha, \lambda} f(x)\|_2^2.$$

□

### 3. VARIATIONAL FORMULATIONS AND SPECTRAL ANALYSIS FOR TEMPERED PROBLEMS

Based on the analysis in Section 2, we can now discuss the weak solutions and spectral methods based on the fractional integral spaces. We focus on the following tempered stationary problems:

$$(3.1) \quad \begin{cases} {}_a D_x^{\alpha_1, \lambda} u(x) + d \cdot {}_a D_x^{\alpha_2, \lambda} u(x) = f(x), & \alpha_1 > \alpha_2 \geq 0, \lambda > 0, \\ {}_a D_x^{\alpha_1 - 1, \lambda} u(x)|_{x=a} = 0, \text{ and } {}_a D_x^{\alpha_1 - 1, \lambda} u(x)|_{x=b} = u_b & (\text{if } 1 < \alpha_1 < 2). \end{cases}$$

where  $u_b \in \mathbb{R}$ . When  $0 < \alpha_1 < 1$ , we call (3.1) a tempered fractional advection problem, while if  $1 < \alpha_2 < \alpha_1 < 2$ , (3.1) is named as tempered fractional diffusion problem. For the case when  $0 < \alpha_2 < 1 < \alpha_1 < 2$ , (3.1) is a tempered fractional advection-diffusion problem, which will be discussed in our future work.

In fact, (3.1) can be generalized to multi-term fractional differential equations [25], which can be likewise be studied without additional difficulty. It has been showed in [11] that multi-term fractional problems are dominated by the behavior of the highest order of fractional operator, which suggests that it suffices to discuss (3.1) to understand general multi-term problems.

Without loss of generality, we restrict our attention to the interval  $(-1, 1)$ .

**3.1. Tempered fractional advection problem.** We begin by discussing (3.1) when  $0 \leq \alpha_2 < \alpha_1 < 1$ .

3.1.1. *Variational formulation.* In order to derive a variational form of (3.1) for  $0 \leq \alpha_2 < \alpha_1 < 1$ , we assume that  $u(x)$  is a sufficiently smooth solution to  ${}_{-1}D_x^{\alpha, \lambda} u(x) = f(x)$ . By multiplying an arbitrary  $v(x) \in C_0^\infty(-1, 1)$ , it can be obtained that

$$\int_{-1}^1 {}_{-1}D_x^{\alpha, \lambda} u(x) \cdot v(x) dx = \int_{-1}^1 {}_{-1}D_x^{\alpha/2, \lambda} u(x) \cdot {}_x D_1^{\alpha/2, \lambda} v(x) dx = \int_{-1}^1 f(x)v(x) dx.$$

Since  $0 \leq \alpha_2 < \alpha_1 < 1$ , by Lemma 2.11, there is  $I^{\alpha_1}[L_2(a, b)] \subseteq I^{\alpha_2}[L_2(a, b)]$ . Thus, by (2.27), we can define the associated bilinear form  $B_1 : I^{\alpha_1/2}[L_2(-1, 1)] \times I^{\alpha_1/2}[L_2(-1, 1)] \rightarrow \mathbb{R}$  for (3.1) as

$$(3.2) \quad B_1(u, v) := \left( {}_{-1}D_x^{\alpha_1/2, \lambda} u(x), {}_x D_1^{\alpha_1/2, \lambda} v(x) \right) + d \left( {}_{-1}D_x^{\alpha_2/2, \lambda} u(x), {}_x D_1^{\alpha_2/2, \lambda} v(x) \right).$$

For a given  $f(x) \in L_p(-1, 1)$ ,  $p = \frac{2}{1+\alpha_1}$ , we define the associated linear functional  $F_1 : I^{\alpha_1/2}[L_2(-1, 1)] \rightarrow \mathbb{R}$  as

$$(3.3) \quad F_1(v) := \langle f, v \rangle.$$

By Lemma 2.9, we know that if  $v(x) \in I^{\alpha_1/2}[L_2(-1, 1)]$ ,  $0 < \alpha_1 < 1$ , then  $v(x) \in L_q(-1, 1)$ ,  $q = \frac{2}{1-\alpha_1}$ . Since  $\frac{1}{p} + \frac{1}{q} = 1$ , the duality pairings in (3.3) are well defined ([13], p.640).

We shall show later that the functions in space  $I^{\alpha_1/2}[L_2(-1, 1)]$  still satisfy the boundary condition in (3.1), i.e.,  ${}_a D_x^{\alpha_1-1, \lambda} u(x) \Big|_{x=-1} = 0$  if  $u(x) \in I^{\alpha_1/2}[L_2(-1, 1)]$ .

Thus, the Galerkin variational solution of (3.1) can be defined as follows.

**Definition 3.1** (Variational Formulation). A function  $u(x) \in I^{\alpha_1/2}[L_2(-1, 1)]$  is a variational solution of problem (3.1) with  $0 \leq \alpha_2 < \alpha_1 < 1$  provided

$$(3.4) \quad B_1(u, v) = F_1(v) \quad \forall v(x) \in I^{\alpha_1/2}[L_2(-1, 1)].$$

We shall now show that there exists a unique solution to (3.4).

**Lemma 3.2.** *The bilinear form  $B_1(\cdot, \cdot)$  is continuous on  $I^{\alpha_1/2}[L_2(-1, 1)] \times I^{\alpha_1/2}[L_2(-1, 1)]$ .*

*Proof.* From the definitions of  $B_1(\cdot, \cdot)$  in (3.2), of the tempered fractional derivatives in Definition 2.2, of (i) of Lemma 2.9, and of the space norm in (2.26), as well as Corollary 2.13, it follows that

$$\begin{aligned} |B_1(u, v)| &\leq \|{}_{-1}D_x^{\alpha_1/2} (e^{\lambda x} u(x))\|_2 \cdot \|{}_x D_1^{\alpha_1/2} (e^{-\lambda x} v(x))\|_2 \\ &\quad + |d| \cdot \|{}_{-1}I_x^{\frac{\alpha_1-\alpha_2}{2}} {}_{-1}D_x^{\alpha_1/2} (e^{\lambda x} u(x))\|_2 \cdot \|{}_x I_1^{\frac{\alpha_1-\alpha_2}{2}} {}_x D_1^{\alpha_1/2} (e^{-\lambda x} v(x))\|_2 \\ &\leq \|e^{\lambda x} u(x)\|_{I^{\alpha_1/2}[L_2(-1, 1)]} \cdot \|e^{-\lambda x} v(x)\|_{I^{\alpha_1/2}[L_2(-1, 1)]} \\ &\leq C \cdot \|u\|_{I^{\alpha_1/2}[L_2(-1, 1)]} \cdot \|v\|_{I^{\alpha_1/2}[L_2(-1, 1)]}. \end{aligned}$$

□

**Lemma 3.3.** *If  $d > -\frac{\Gamma^2(\frac{\alpha_1-\alpha_2}{2}+1)}{2^{\alpha_1-\alpha_2}}$ , then the bilinear form  $B_1(\cdot, \cdot)$  satisfies the inf-sup condition, i.e.,  $\exists C > 0$ , such that*

$$(3.5) \quad \sup_{0 \neq v \in I^{\frac{\alpha_1}{2}}(L_2)} \frac{|B_1(u, v)|}{\|u\|_{I^{\frac{\alpha_1}{2}}(L_2)} \cdot \|v\|_{I^{\frac{\alpha_1}{2}}(L_2)}} \geq C \quad \forall 0 \neq u \in I^{\frac{\alpha_1}{2}}(L_2).$$

*Proof.* For any  $u(x) \in I^{\alpha_1/2}[L_2(-1, 1)]$ , there exists  $\phi(x) \in L_2(-1, 1)$ , such that  $e^{\lambda x} u(x) = {}_{-1}I_x^{\frac{\alpha_1}{2}} \phi(x)$ . Let  $v(x) = e^{\lambda x} {}_x I_1^{\frac{\alpha_1}{2}} \phi(x) \in {}_x I_1^{\frac{\alpha_1}{2}}[L_2(-1, 1)]$ . Then, if  $d < 0$ , by Lemma 2.9 and Corollary 2.13, we have

$$\begin{aligned}
(3.6) \quad |B_1(u, v)| &\geq \|\phi(x)\|_2^2 + d \cdot \left\| {}_{-1}I_x^{\frac{\alpha_1 - \alpha_2}{2}} \phi(x) \right\|_2 \cdot \left\| {}_x I_1^{\frac{\alpha_1 - \alpha_2}{2}} \phi(x) \right\|_2 \\
&\geq (1 + d \cdot 2^{\alpha_1 - \alpha_2} / \Gamma^2(\frac{\alpha_1 - \alpha_2}{2} + 1)) \cdot \|\phi(x)\|_2^2 \\
&\geq C \cdot \left\| e^{-\lambda x} {}_{-1}I_x^{\alpha_1/2} \phi(x) \right\|_{I^{\frac{\alpha_1}{2}}[L_2(-1, 1)]} \cdot \left\| e^{\lambda x} {}_x I_1^{\alpha_1/2} \phi(x) \right\|_{I^{\frac{\alpha_1}{2}}[L_2(-1, 1)]} \\
&= C \cdot \|u\|_{I^{\frac{\alpha_1}{2}}[L_2(-1, 1)]} \cdot \|v\|_{I^{\frac{\alpha_1}{2}}[L_2(-1, 1)]}.
\end{aligned}$$

While if  $d \geq 0$ , we can get similar result by using Lemma 2.18. So,

$$(3.7) \quad \sup_{0 \neq v \in I^{\frac{\alpha_1}{2}}(L_2)} \frac{|B_1(u, v)|}{\|u\|_{I^{\frac{\alpha_1}{2}}(L_2)} \cdot \|v\|_{I^{\frac{\alpha_1}{2}}(L_2)}} \geq C \forall 0 \neq u \in I^{\frac{\alpha_1}{2}}(L_2).$$

□

Similarly, we can prove the following lemma.

**Lemma 3.4.** *If  $d > -\frac{\Gamma^2(\frac{\alpha_1 - \alpha_2}{2} + 1)}{2^{\alpha_1 - \alpha_2}}$ , then the bilinear form  $B_1(\cdot, \cdot)$  satisfies the “transposed” inf-sup condition, i.e.,*

$$(3.8) \quad \sup_{0 \neq u \in I^{\frac{\alpha_1}{2}}(L_2)} |B_1(u, v)| > 0 \forall 0 \neq v \in I^{\frac{\alpha_1}{2}}(L_2).$$

**Lemma 3.5.** *The linear function  $F_1(\cdot)$  is continuous over  $I^{\alpha_1/2}[L_2(-1, 1)]$ .*

*Proof.* This result follows from Lemma 2.9 since

$$F_1(v) = \langle f, v \rangle \leq \|f\|_p \cdot \|v\|_q \leq C \|f\|_p \cdot \|v\|_{I^{\alpha_1/2}[L_2(-1, 1)]},$$

where  $p = \frac{2}{1 + \alpha_1}$ ,  $q = \frac{2}{1 - \alpha_1}$ , and  $\frac{1}{p} + \frac{1}{q} = 1$ . □

By Lemmas 3.2-3.5,  $B_1(\cdot, \cdot)$  and  $F_1(\cdot)$  satisfy the hypotheses of the inf-sup condition [2]. Thus, the weak formulation (3.4) has a unique solution.

**Theorem 3.6.** *If  $d > -\frac{\Gamma^2(\frac{\alpha_1 - \alpha_2}{2} + 1)}{2^{\alpha_1 - \alpha_2}}$ , then the variation formulation (3.4) admits a unique solution  $u(x) \in I^{\alpha_1/2}[L_2(-1, 1)]$  satisfying*

$$(3.9) \quad \|u(x)\|_2 \leq C \|u(x)\|_q \leq C \|u(x)\|_{I^{\alpha_1/2}[L_2(-1, 1)]} \leq C \|f\|_p,$$

where  $p = \frac{2}{1 + \alpha_1}$ ,  $q = \frac{2}{1 - \alpha_1}$ .

*Proof.* Since  $u(x) \in I^{\alpha_1/2}[L_2(-1, 1)]$ ,  $0 < \alpha_1 < 1$ , by Lemma 2.9, we know that  $u(x)$  is bounded from  $I^{\alpha_1/2}[L_2(-1, 1)]$  into  $L_q(-1, 1)$ .

Since  $q = \frac{2}{1 - \alpha_1} > 2$ , and  $L_q(-1, 1) \subseteq L_2(-1, 1)$ , we have  $\|u(x)\|_2 \leq C \|u(x)\|_q$ . □

**3.1.2. Petrov-Galerkin spectral methods.** In this subsection we propose a spectral method to numerically solve (3.4).

Define  $P_{N-1}[-1, 1]$  as the polynomials spaces of degree less than or equal to  $N - 1$  (if  $N = 0$ , we define  $P_{-1}[-1, 1] := \{0\}$ ), and define

$$(3.10) \quad \Phi_N[-1, 1] := e^{-\lambda x} \cdot {}_{-1}I_x^{\alpha_1/2} [P_{N-1}[-1, 1]], \text{ and } \Psi_N[-1, 1] := e^{\lambda x} \cdot {}_x I_1^{\alpha_1/2} [P_{N-1}[-1, 1]].$$

It is clear from Corollary 2.13 that  $\Phi_N[-1, 1] \subseteq I^{\alpha_1/2}[L_2(-1, 1)]$ ,  $\Psi_N[-1, 1] \subseteq I^{\alpha_1/2}[L_2(-1, 1)]$ .

The Petrov-Galerkin approximation of (3.4) is: find  $u_N(x) \in \Phi_N[-1, 1]$ , such that

$$(3.11) \quad B_1(u_N, \psi_N) = \langle f, \psi_N \rangle \quad \forall \psi_N(x) \in \Psi_N[-1, 1].$$

Since  $\Phi_N[-1, 1]$  is not the same as  $\Psi_N[-1, 1]$ , to prove the existence and uniqueness of the solution to (3.11), we need to show [2]:

$$(3.12) \quad \bullet \text{ Discrete inf-sup condition: } \exists C > 0, \text{ such that}$$

$$\sup_{0 \neq \psi_N \in \Psi_N} \frac{|B_1(u_N, \psi_N)|}{\|u_N\|_{I^{\alpha_1/2}[L_2(-1, 1)]} \cdot \|\psi_N\|_{I^{\alpha_1/2}[L_2(-1, 1)]}} \geq C \quad \forall 0 \neq u_N \in \Phi_N[-1, 1],$$

$$\bullet \text{ Discrete "transposed" inf-sup condition:}$$

$$(3.13) \quad \sup_{0 \neq u_N \in \Phi_N[-1, 1]} |B_1(u_N, \psi_N)| > 0 \quad \forall 0 \neq \psi_N \in \Psi_N[-1, 1].$$

These can be proved similarly to Lemmas 3.3 and 3.5. Thus, the following result holds.

**Theorem 3.7.** *If  $d > -\frac{\Gamma^2(\frac{\alpha_1 - \alpha_2}{2} + 1)}{2^{\alpha_1 - \alpha_2}}$ , then the discrete problem (3.11) admits a unique solution  $u_N(x) \in \Phi_N[-1, 1]$ , satisfying*

$$(3.14) \quad \|u_N(x)\|_{I^{\alpha_1/2}[L_2(-1, 1)]} \leq C \|f(x)\|_p.$$

Moreover, if  $u(x)$  is the solution of (3.4), we have

$$(3.15) \quad \|u - u_N\|_2 \leq C \|u - u_N\|_q \leq C \|u - u_N\|_{I^{\alpha_1/2}[L_2(-1, 1)]} \leq C \inf_{\phi_N \in \Phi_N} \|u - \phi_N\|_{I^{\alpha_1/2}[L_2(-1, 1)]},$$

$$\text{where } p = \frac{2}{1 + \alpha_1}, \quad q = \frac{2}{1 - \alpha_1}.$$

**3.1.3. Error estimates.** Denote  $\Pi_N$  as the  $L_2(-1, 1)$  orthogonal projection operator. For  $\Pi_N$ , the error estimate is well known ([3], p.283):

$$(3.16) \quad \|v(x) - \Pi_N v(x)\|_2 \leq CN^{-m} |v(x)|_{H^m(-1, 1)} \quad \forall v(x) \in H^m(-1, 1), \quad m \geq 0,$$

where  $H^m(-1, 1) = W^{m, 2}(-1, 1)$  consists of all locally summable functions  $\varphi : (-1, 1) \rightarrow \mathbb{R}$ , such that for each  $k$ ,  $0 \leq k \leq m$ ,  $D^k \varphi(x)$  exists in the weak sense and belongs to  $L_2(-1, 1)$ .

We can combine the previous results into an error estimate.

**Lemma 3.8.** *Let  $u(x)$  solve (3.4), and  $u_N(x)$  solve (3.11). If  $e^{\lambda x} u(x) \in {}_{-1}I_x^{\alpha_1/2}[H^m(-1, 1)]$ ,  $m \geq 0$ , then there is a constant  $C$  such that*

$$(3.17) \quad \|u(x) - u_N(x)\|_2 \leq CN^{-m} \left\| {}_{-1}D_x^{m + \alpha_1/2} (e^{\lambda x} u(x)) \right\|_2.$$

*Proof.* By Lemma 2.7 we know that if  $e^{\lambda x} u(x) \in {}_{-1}I_x^{\alpha_1/2}[H^m(-1, 1)]$ , then  ${}_{-1}D_x^{\alpha_1/2}(e^{\lambda x} u(x)) \in H^m(-1, 1)$ . So, from (3.15), the definition (2.26), and Theorem 3.7, we

have

$$\begin{aligned}
(3.18) \quad \|u(x) - u_N(x)\|_2 &\leq C \inf_{v_N \in \Phi_N[-1,1]} \|e^{\lambda x} u(x) - e^{\lambda x} v_N(x)\|_{I_x^{\alpha_1/2}[L_2(-1,1)]} \\
&\leq C \left\| {}_{-1}D_x^{\alpha_1/2} (e^{\lambda x} u(x)) - \Pi_N \left( {}_{-1}D_x^{\alpha_1/2} (e^{\lambda x} u(x)) \right) \right\|_2 \\
&\leq CN^{-m} \left\| {}_{-1}D_x^{m+\alpha_1/2} (e^{\lambda x} u(x)) \right\|_2.
\end{aligned}$$

□

We have the following result to describe the regularity of the solution to (3.4). The proof of this is given in Appendix B.

**Theorem 3.9.** *Let  $u(x)$  solve (3.1) with  $0 \leq \alpha_2 < \alpha_1 < 1$ , then the following statements are equivalent:*

(i)

$$(3.19) \quad e^{\lambda x} u(x) \in {}_{-1}I_x^{\alpha_1/2}[H^m(-1,1)], \quad m \geq 0;$$

(ii)

$$(3.20) \quad u(x) = e^{-\lambda x} (u_1(x) + u_2(x)),$$

where

$$(3.21a) \quad u_1(x) \in (1+x)^{\alpha_1/2} \cdot P_{m-1}[-1,1],$$

(3.21b)

$$u_2(x) \in W^{m,q}(-1,1), \quad q = \frac{2}{1-\alpha_1}, \quad \text{and } u_2(x) = o\left((1+x)^{m+\frac{\alpha_1}{2}-\frac{1}{2}}\right), \quad \text{as } x \rightarrow -1;$$

(iii)

$$(3.22) \quad f(x) = e^{-\lambda x} (f_1(x) + f_2(x)),$$

where

$$(3.23a) \quad f_1(x) \in (1+x)^{-\alpha_1/2} \cdot P_{m-1}[-1,1] + (1+x)^{\alpha_1/2-\alpha_2} \cdot P_{m-1}[-1,1],$$

(3.23b)

$$f_2(x) \in W^{m,p}(-1,1), \quad p = \frac{2}{1+\alpha_1}, \quad \text{and } f_2(x) = o\left((1+x)^{m-\frac{\alpha_1}{2}-\frac{1}{2}}\right), \quad \text{as } x \rightarrow -1.$$

*Remark 3.10.* Here we note that from (3.20)-(3.21b), it can be seen  ${}_{-1}D_x^{\alpha_1-1,\lambda} u(x)|_{x=-1} = 0$  holds, i.e.,  $u(x)$  satisfies the boundary conditions of (3.1) for  $0 < \alpha_1 < 1$ .

*Remark 3.11.* Unlike the result in Lemma 2.11, where  $e^{\lambda x} u(x) \in {}_{-1}I_x^{\alpha_1/2}[L_2(-1,1)]$  implies  $e^{\lambda x} u(x) \in {}_{-1}I_x^{\alpha_2/2}[L_2(-1,1)]$ , here  $e^{\lambda x} u(x) \in {}_{-1}I_x^{\alpha_2/2}[H^m(-1,1)]$  cannot be derived from  $e^{\lambda x} u(x) \in {}_{-1}I_x^{\alpha_1/2}[H^m(-1,1)]$ , when  $m \geq 1$ . For example, for a constant function  $c_0$ , it is clear that  ${}_{-1}I_x^{\alpha_1/2} c_0 = \frac{c_0}{\Gamma(1+\alpha_1/2)} (x+1)^{\alpha_1/2} \in {}_{-1}I_x^{\alpha_1/2}[H^m(-1,1)]$ , and  $\frac{c_0}{\Gamma(1+\alpha_1/2)} (x+1)^{\alpha_1/2} = \frac{c_0}{\Gamma(1+\frac{\alpha_1-\alpha_2}{2})} {}_{-1}I_x^{\alpha_2/2} (x+1)^{\frac{\alpha_1-\alpha_2}{2}}$ . Since  $(x+1)^{\frac{\alpha_1-\alpha_2}{2}} \notin H^m(-1,1)$ ,  ${}_{-1}I_x^{\alpha_1/2} c_0 \notin {}_{-1}I_x^{\alpha_2/2}[H^m(-1,1)]$ .

Combining Lemma 3.8 and Theorem 3.9, we recover the error estimate as follows.

**Theorem 3.12.** Let  $e^{-\lambda x}u(x) \in {}_{-1}I_x^{\alpha_1/2}[H^m(-1,1)]$ ,  $m \geq 0$ , solve (3.4), and  $u_N(x)$  solve (3.11). If  $|d| < \frac{\Gamma(\alpha_1 - \alpha_2 + 1)}{2^{\alpha_1 - \alpha_2}}$ , then, there is a constant  $C$  such that

$$(3.24) \quad \|u(x) - u_N(x)\|_2 \leq CN^{-m} \|f_2(x)\|_{W^{m,p}(-1,1)},$$

where  $p = \frac{2}{1+\alpha_1}$ , and  $f_2(x)$  is given in (3.23b).

*Proof.* If  $e^{\lambda x}u(x) \in {}_{-1}I_x^{\alpha_1/2}[H^m(-1,1)]$ , by the above analysis we have

$$(3.25) \quad D^m {}_{-1}D_x^{\alpha_1/2} (e^{\lambda x}u(x)) = {}_{-1}I_x^{\alpha_1/2} D^m f_2(x) - d \cdot {}_{-1}I_x^{\alpha_1 - \alpha_2} D^m {}_{-1}D_x^{\alpha_1/2} (e^{\lambda x}u(x)),$$

which implies by Lemma 2.9 that

$$(3.26) \quad \begin{aligned} & \left\| D^m {}_{-1}D_x^{\alpha_1/2} (e^{\lambda x}u(x)) \right\|_2 \\ & \leq C \|D^m f_2(x)\|_p + |d| \cdot \frac{2^{\alpha_1 - \alpha_2}}{\Gamma(\alpha_1 - \alpha_2 + 1)} \left\| D^m {}_{-1}D_x^{\alpha_1/2} (e^{\lambda x}u(x)) \right\|_2. \end{aligned}$$

Thus,

$$(3.27) \quad \left( 1 - |d| \cdot \frac{2^{\alpha_1 - \alpha_2}}{\Gamma(\alpha_1 - \alpha_2 + 1)} \right) \left\| D^m {}_{-1}D_x^{\alpha_1/2} (e^{\lambda x}u(x)) \right\|_2 \leq C \|f_2(x)\|_{W^{m,p}(-1,1)}.$$

□

*Remark 3.13.* We note that although the limitation for the parameter  $d$  is required for the theoretical analysis, the examples in Section 4.2 suggest that it can be relaxed.

3.1.4. *Numerical implementation.* In order to numerically solve (3.4), we recall the following formulas ([1], p.20):

$$(3.28) \quad {}_{-1}I_x^\alpha ((1+x)^\delta J_n^{\gamma,\delta}(x)) = \frac{\Gamma(n+\delta+1)}{\Gamma(n+\delta+\alpha+1)} (1+x)^{\delta+\alpha} J_n^{\gamma-\alpha,\delta+\alpha}(x),$$

$$(3.29) \quad {}_xI_1^\alpha ((1-x)^\delta J_n^{\delta,\gamma}(x)) = \frac{\Gamma(n+\delta+1)}{\Gamma(n+\delta+\alpha+1)} (1-x)^{\delta+\alpha} J_n^{\delta+\alpha,\gamma-\alpha}(x),$$

where  $\alpha > 0$ ,  $\gamma > -1$ ,  $\beta \in \mathbb{R}$ , and the Jacobi polynomials,  $J_n^{\alpha,\beta}(x)$ , are defined by

$$(3.30) \quad (1-x)^\alpha (1+x)^\beta J_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1-x)^{n+\alpha} (1+x)^{n+\beta}].$$

Recall that Jacobi polynomials are orthogonal on  $[-1, 1]$  with respect to  $(1-x)^\alpha (1+x)^\beta$  when  $\alpha > -1$ ,  $\beta > -1$ , however, many of the formulas also hold when  $\alpha \in \mathbb{R}$  or  $\beta \in \mathbb{R}$  [27], e.g., Eqs. (3.28) and (3.29), and

$$(3.31) \quad J_n^{\alpha,\beta}(x) = (-1)^n J_n^{\beta,\alpha}(-x),$$

$$(3.32) \quad J_n^{\alpha,\beta}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

Using the properties of the Riemann-Liouville fractional derivatives  ${}_{-1}D_x^\alpha {}_{-1}I_x^\alpha = I$  and  ${}_x D_1^\alpha {}_x I_1^\alpha = I$ , formulae (3.28) and (3.29) can be rewritten, respectively, as

$$(3.33) \quad {}_{-1}D_x^\alpha ((1+x)^{\delta+\alpha} J_n^{\gamma-\alpha,\delta+\alpha}(x)) = \frac{\Gamma(n+\delta+\alpha+1)}{\Gamma(n+\delta+1)} (1+x)^\delta J_n^{\gamma,\delta}(x),$$

$$(3.34) \quad {}_x D_1^\alpha \left( (1-x)^{\delta+\alpha} J_n^{\delta+\alpha, \gamma-\alpha}(x) \right) = \frac{\Gamma(n+\delta+\alpha+1)}{\Gamma(n+\delta+1)} (1-x)^\delta J_n^{\delta, \gamma}(x).$$

*Remark 3.14.* It is clear from Eqs. (3.28), (3.29), and (3.33), (3.34) that  ${}_{-1}I_x^\alpha {}_{-1}D_x^\alpha = I$  and  ${}_x I_1^\alpha {}_x D_1^\alpha = I$  hold for functions  $(1+x)^{\delta+\alpha} J_n^{\gamma-\alpha, \delta+\alpha}(x)$  and  $(1-x)^{\delta+\alpha} J_n^{\delta+\alpha, \gamma-\alpha}(x)$ , respectively, when  $\alpha > 0$ ,  $\delta > -1$ , and  $\gamma \in \mathbb{R}$ .

We construct trial functions and test functions respectively as follows:

$$(3.35) \quad \phi_n(x) := e^{-\lambda x} {}_{-1}I_x^{\frac{\alpha_1}{2}} L_n(x) = \frac{\Gamma(n+1)}{\Gamma(n+1+\frac{\alpha_1}{2})} e^{-\lambda x} (1+x)^{\frac{\alpha_1}{2}} J_n^{-\frac{\alpha_1}{2}, \frac{\alpha_1}{2}}(x), \quad 0 \leq n \leq N-1,$$

and

$$(3.36) \quad \psi_n(x) := e^{\lambda x} {}_x I_1^{\frac{\alpha_1}{2}} L_n(x) = \frac{\Gamma(n+1)}{\Gamma(n+1+\frac{\alpha_1}{2})} e^{\lambda x} (1-x)^{\frac{\alpha_1}{2}} J_n^{\frac{\alpha_1}{2}, -\frac{\alpha_1}{2}}(x), \quad 0 \leq n \leq N-1,$$

where  $L_n(x) = J_n^{0,0}(x)$ ,  $n \geq 0$ , are Legendre polynomials, which are orthogonal in the  $L_2$  sense [3, 15]:

$$(3.37) \quad \int_{-1}^1 L_n(x) L_m(x) dx = \gamma_n \delta_{mn}, \quad \gamma_n = \frac{2}{2n+1}.$$

Denote

$$(3.38) \quad u_N(x) = \sum_{n=0}^{N-1} \hat{u}_n \phi_n(x) = e^{-\lambda x} \sum_{n=0}^{N-1} \hat{u}_n \frac{\Gamma(n+1)}{\Gamma(n+1+\frac{\alpha_1}{2})} (1+x)^{\frac{\alpha_1}{2}} J_n^{-\frac{\alpha_1}{2}, \frac{\alpha_1}{2}}(x).$$

By Eqs. (3.2), (3.3), (3.33), (3.34), and (3.37), (3.11) can be rewritten as

$$(3.39) \quad \mathbf{A} \mathbf{u} = \mathbf{f},$$

where  $\mathbf{u} = [\hat{u}_0, \hat{u}_1, \dots, \hat{u}_{N-1}]^T$ ,  $\mathbf{f}_k = \int_{-1}^1 f(x) \psi_k(x) dx$ ,  $k = 0, 1, \dots, N-1$ ,  $\mathbf{A} = \mathbf{A}_1 + d \cdot \mathbf{A}_2$ , and

$$(3.40) \quad (\mathbf{A}_1)_{n,n} = \frac{2}{2n+1}, \quad (\mathbf{A}_1)_{m,n} = 0, \quad m \neq n, \quad n = 0, 1, \dots, N-1.$$

(3.41)

$$(\mathbf{A}_2)_{k,n} = \frac{\Gamma(k+1)\Gamma(n+1)}{\Gamma(k+1+\frac{\alpha_1-\alpha_2}{2})\Gamma(n+1+\frac{\alpha_1-\alpha_2}{2})} \int_{-1}^1 (1-x^2)^{\frac{\alpha_1-\alpha_2}{2}} J_k^{\frac{\alpha_1-\alpha_2}{2}, -\frac{\alpha_1-\alpha_2}{2}}(x) J_n^{-\frac{\alpha_1-\alpha_2}{2}, \frac{\alpha_1-\alpha_2}{2}}(x) dx, \quad k, n = 0, 1, \dots, N-1.$$

*Remark 3.15.* By Eq. (3.31), we have

$$(3.42) \quad \begin{aligned} & \int_{-1}^1 (1-x^2)^{\frac{\alpha_1-\alpha_2}{2}} J_k^{\frac{\alpha_1-\alpha_2}{2}, -\frac{\alpha_1-\alpha_2}{2}}(x) J_n^{-\frac{\alpha_1-\alpha_2}{2}, \frac{\alpha_1-\alpha_2}{2}}(x) dx \\ &= (-1)^{n+k} \sum_{i=0}^N J_n^{\frac{\alpha_1-\alpha_2}{2}, -\frac{\alpha_1-\alpha_2}{2}}(-x_i) J_k^{-\frac{\alpha_1-\alpha_2}{2}, \frac{\alpha_1-\alpha_2}{2}}(-x_i) \omega_i \\ &= (-1)^{n+k} \sum_{i=0}^N J_n^{\frac{\alpha_1-\alpha_2}{2}, -\frac{\alpha_1-\alpha_2}{2}}(x_{N-i}) J_k^{-\frac{\alpha_1-\alpha_2}{2}, \frac{\alpha_1-\alpha_2}{2}}(x_{N-i}) \omega_i \\ &= (-1)^{n+k} \sum_{i=0}^N J_n^{\frac{\alpha_1-\alpha_2}{2}, -\frac{\alpha_1-\alpha_2}{2}}(x_i) J_k^{-\frac{\alpha_1-\alpha_2}{2}, \frac{\alpha_1-\alpha_2}{2}}(x_i) \omega_i, \end{aligned}$$

where  $\{x_i\}_{i=0}^N$  and  $\{\omega_i\}_{i=0}^N$  are Gauss quadrature nodes and wights w.r.t. the weight function  $(1-x)^{\frac{\alpha_1-\alpha_2}{2}}(1+x)^{\frac{\alpha_1-\alpha_2}{2}}$ , respectively. Since  $-x_i = x_{N-i}$ , because the symmetric weight function [3], and we have

$$(3.43) \quad (\mathbf{A}_2)_{k,n} = (-1)^{n+k} (\mathbf{A}_2)_{n,k}.$$



*Remark 3.16.* Here we emphasize that if  $u(x) \in {}_{-1}D_x^{\alpha/2}[L_2(-1, 1)]$ , then  $u(x) \in L_q(-1, 1)$ ,  $q = \frac{2}{1-\alpha}$ , and  ${}_{-1}D_x^{\alpha/2}u(x) \in L_2(-1, 1)$ , so  ${}_{-1}D_x^{\alpha/2}u(x) = \sum_{n=0}^{\infty} \hat{u}_n L_n(x)$ , where  $\hat{u}_n = \frac{2n+1}{2} \int_{-1}^1 {}_{-1}D_x^{\alpha/2}u(x)L_n(x) dx$ . Also  $\|u(x) - u_N(x)\|_q \rightarrow 0$ , as  $N \rightarrow \infty$ , see [27]. In particular, we can see in numerical results that if  ${}_{-1}D_x^{\alpha/2}u(x)$  being as a jump function, then there is even  $\|u(x) - u_N(x)\|_q \leq CN^{-1/2} \cdot \|{}_{-1}D_x^{\alpha/2}u(x)\|_2$ . That is, although  $u_N(-1) \equiv 0$ , we can still approximate the function  $u(x)$ , which might be **non-zero or even unbounded** at the left boundary point, in the sense of  $L_q(-1, 1)$ .

**3.2. Tempered fractional diffusion problem.** Let us not finally discuss (3.1) when  $1 < \alpha_2 < \alpha_1 < 2$ .

**3.2.1. Variational formulation.** In order to derive a variational form of problem (3.1) for  $1 < \alpha_2 < \alpha_1 < 2$ , we assume  $u(x)$  is sufficiently smooth, and  $v(x) \in C_0^\infty(-1, 1)$ . Then, for  $1 < \alpha < 2$

$$\begin{aligned} \int_{-1}^1 {}_{-1}D_x^{\alpha, \lambda}u(x) \cdot v(x) dx &= \int_{-1}^1 {}_{-1}D_x^{1, \lambda} {}_{-1}D_x^{\alpha-1, \lambda}u(x) \cdot v(x) dx \\ &= \int_{-1}^1 {}_{-1}D_x^{\frac{\alpha-1}{2}, \lambda}u(x) \cdot {}_x D_1^{\frac{\alpha+1}{2}, \lambda}v(x) dx. \end{aligned}$$

Since  $v(1) = 0$  if  $v(x) \in {}_x I_1^{\frac{\alpha+1}{2}}[L_2(-1, 1)]$  when  $\alpha > 1$  by (2.18), we can define the Petrov-Galerkin variational formulation of (3.1) for  $1 < \alpha_2 < \alpha_1 < 2$  as follows.

Let  
(3.44)

$$X(-1, 1) := \left\{ u : u(x) \in {}_{-1}I_x^{\frac{\alpha_1-1}{2}}[L_2(-1, 1)], \text{ and } {}_{-1}D_x^{\alpha_1-1}(e^{\lambda x}u(x)) \Big|_{x=-1} = 0 \right\}.$$

**Definition 3.17** (Variational Formulation). A function  $u(x) \in X(-1, 1)$  is a variational solution of problem (3.1) with  $1 < \alpha_2 < \alpha_1 < 2$ , provided  ${}_{-1}D_x^{\alpha_1-1}u(x) \Big|_{x=1} = u_b$ , and

$$(3.45) \quad B_2(u, v) = F_2(v) \quad \forall v(x) \in {}_x I_1^{\frac{\alpha_1+1}{2}}[L_2(-1, 1)],$$

where the bilinear form  $B_2 : X(-1, 1) \times {}_x I_1^{\frac{\alpha_1+1}{2}}[L_2(-1, 1)] \rightarrow \mathbb{R}$  is defined as  
(3.46)

$$B_2(u, v) := \left( {}_{-1}D_x^{\frac{\alpha_1-1}{2}, \lambda}u(x), {}_x D_1^{\frac{\alpha_1+1}{2}, \lambda}v(x) \right) + d \left( {}_{-1}D_x^{\frac{\alpha_2-1}{2}, \lambda}u(x), {}_x D_1^{\frac{\alpha_2+1}{2}, \lambda}v(x) \right),$$

and for a given  $f(x) \in W^{-1, p}(-1, 1)$ ,  $p = \frac{2}{\alpha_1}$ , and  $f(x) = o((x+1)^{-1})$  as  $x \rightarrow -1$ , the linear functional  $F_2 : {}_x I_1^{\frac{\alpha_1+1}{2}}[L_2(-1, 1)] \rightarrow \mathbb{R}$  as

$$(3.47) \quad F_2(v) := \langle f, v \rangle.$$

We first prove that there is a unique solution to (3.45).

**Lemma 3.18.** *The linear function  $F_2(\cdot)$  is continuous over  ${}_x I_1^{\frac{\alpha_1+1}{2}}[L_2(-1, 1)]$ .*

*Proof.* If  $v(x) \in {}_x I_1^{\frac{\alpha_1+1}{2}}[L_2(-1, 1)]$ , then  $v(x) \in W^{1, q}(-1, 1)$ , where  $q = \frac{2}{2-\alpha_1}$ . Since  $\frac{1}{2} < \frac{\alpha_1+1}{2} < 1 + \frac{1}{2}$ , by (iv) of Lemma 2.9, there is  $v(x) \in C^0, \frac{\alpha_1}{2}[-1, 1]$ . Thus

$$\begin{aligned} \int_{-1}^1 |Dv(x)|^q dx &= \int_{-1}^1 \lim_{y \rightarrow x} \left| \frac{v(y)-v(x)}{(y-x)^{\frac{\alpha_1}{2}}} \cdot (y-x)^{\frac{\alpha_1-2}{2}} \right|^q dx \\ (3.48) \quad &= \int_{-1}^1 \lim_{y \rightarrow x} \left| o((y-x)^{-\frac{1}{q}}) \right|^q dx < \infty. \end{aligned}$$

Furthermore, since  $v(1) = 0$  by (2.18), and  ${}_{-1}I_x^1 f(x)|_{x=-1} = 0$  by  $f(x) = o((1+x)^{-1})$ , and  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$(3.49) \quad \begin{aligned} |\langle f, v \rangle| &= \left| \int_{-1}^1 {}_{-1}I_x^1 f(x) \cdot Dv(x) dx \right| \\ &\leq \left\| {}_{-1}I_x^1 f(x) \right\|_p \cdot \|Dv(x)\|_q \leq C \|f(x)\|_{W^{-1,p}(-1,1)} \cdot \|v(x)\|_{xI_1^{\frac{\alpha_1+1}{2}}[L_2(-1,1)]}. \end{aligned}$$

□

**Lemma 3.19.** *The bilinear form  $B_2(\cdot, \cdot)$  is continuous on  $X(-1, 1) \times xI_1^{\frac{\alpha_1+1}{2}}[L_2(-1, 1)]$ .*

The proof of Lemma 3.19 is similar to that of Lemma 3.2, and we omit it.

**Lemma 3.20.** *If  $d > -\Gamma^2(\frac{\alpha_1-\alpha_2}{2} + 1)/2^{\alpha_1-\alpha_2}$ , then the bilinear form  $B_2(\cdot, \cdot)$  satisfies the inf-sup condition.*

*Proof.* We only prove it for the case of  $-\Gamma^2(\frac{\alpha_1-\alpha_2}{2} + 1)/2^{\alpha_1-\alpha_2} < d < 0$ . For any  $u(x) \in I^{\frac{\alpha_1-1}{2}}[L_2(-1, 1)]$ , there exists  $\phi(x) \in L_2(-1, 1)$ , such that  $e^{\lambda x}u(x) = {}_{-1}I_x^{\frac{\alpha_1-1}{2}}\phi(x)$ . Let  $v(x) = e^{\lambda x}xI_1^{\frac{\alpha_1+1}{2}}\phi(x) \in xI_1^{\frac{\alpha_1+1}{2}}[L_2(-1, 1)] \subseteq xI_1^{\frac{\alpha_2+1}{2}}[L_2(-1, 1)]$ . Then, by Lemma 2.9, we have

$$(3.50) \quad \begin{aligned} |B_2(u, v)| &\geq \|\phi(x)\|_2^2 + d \cdot \left| \left( {}_{-1}I_x^{\frac{\alpha_1-\alpha_2}{2}}\phi(x), xI_1^{\frac{\alpha_1-\alpha_2}{2}}\phi(x) \right) \right| \\ &\geq \left( 1 + d \cdot \frac{2^{\alpha_1-\alpha_2}}{\Gamma^2(\frac{\alpha_1-\alpha_2}{2} + 1)} \right) \cdot \|e^{\lambda x}u(x)\|_{-1I_x^{\frac{\alpha_1-1}{2}}(L_2)} \cdot \|e^{-\lambda x}v(x)\|_{xI_1^{\frac{\alpha_1+1}{2}}(L_2)} \\ &\geq C \cdot \|u(x)\|_{-1I_x^{\frac{\alpha_1-1}{2}}(L_2)} \cdot \|v(x)\|_{xI_1^{\frac{\alpha_1+1}{2}}(L_2)}. \end{aligned}$$

Thus,

$$(3.51) \quad \sup_{0 \neq v \in xI_1^{\frac{\alpha_1+1}{2}}(L_2)} \frac{|B_2(u, v)|}{\|u\|_{-1I_x^{\frac{\alpha_1-1}{2}}(L_2)} \cdot \|v\|_{xI_1^{\frac{\alpha_1+1}{2}}(L_2)}} \geq C \forall 0 \neq u \in {}_{-1}I_x^{\frac{\alpha_1-1}{2}}(L_2),$$

□

**Lemma 3.21.** *If  $|d| < \frac{\Gamma^2(\frac{\alpha_1-\alpha_2}{2} + 1)}{2^{\alpha_1-\alpha_2+1}}$ , then the bilinear form  $B_2(\cdot, \cdot)$  satisfies the “transposed” inf-sup condition.*

*Proof.* Since  $\{L_n(x)\}_{n=0}^\infty$  is complete in the  $L_2(-1, 1)$  space, we prove that for any  $n \geq 0$ , and  $e^{-\lambda}v(x) := xI_1^{\frac{\alpha_1+1}{2}}L_n(x)$ , there exists  $u(x) \in X(-1, 1)$ , such that  $|B_2(u, v)| > 0$ .

Let

$$(3.52) \quad u(x) := e^{-\lambda x} {}_{-1}I_x^{\frac{\alpha_1-1}{2}}(L_n(x) + L_{n+1}(x)) \in {}_{-1}I_x^{\frac{\alpha_1-1}{2}}[L_2(-1, 1)].$$

By Eq. (3.33) and the property of Jacobi Polynomials, we have

$$(3.53) \quad \begin{aligned} {}_{-1}D_x^{\alpha_1-1, \lambda}u(x) &= e^{-\lambda x} \frac{\Gamma(n+2)}{\Gamma(n+2-\frac{\alpha_1-1}{2})} (1+x)^{-\frac{\alpha_1-1}{2}} \cdot \\ &\quad \left( \left( 1 - \frac{\alpha_1-1}{2(n+1)} \right) J_n^{\frac{\alpha_1-1}{2}, -\frac{\alpha_1-1}{2}}(x) + J_{n+1}^{\frac{\alpha_1-1}{2}, -\frac{\alpha_1-1}{2}}(x) \right) \\ &= e^{-\lambda x} \frac{\Gamma(n+2)}{\Gamma(n+2-\frac{\alpha_1-1}{2})} (1+x)^{\frac{3-\alpha_1}{2}} J_n^{\frac{\alpha_1-1}{2}, \frac{3-\alpha_1}{2}}(x). \end{aligned}$$

Thus  ${}_{-1}D_x^{\alpha_1-1, \lambda} u(x)|_{x=-1} = 0$ . Then, by Lemma 2.9, we have

$$\begin{aligned}
(3.54) \quad & |B_2(u, v)| \\
& \geq \|L_n(x)\|_2^2 - |d| \cdot \left| \left( {}_{-1}I_x^{\frac{\alpha_1-\alpha_2}{2}} (L_n(x) + L_{n+1}(x)), {}_xI_1^{\frac{\alpha_1-\alpha_2}{2}} L_n(x) \right) \right| \\
& \geq \frac{1}{2} \|L_n(x)\|_2 \cdot \|L_n(x) + L_{n+1}(x)\|_2 - |d| \cdot \frac{2^{\alpha_1-\alpha_2}}{\Gamma^2(\frac{\alpha_1-\alpha_2}{2}+1)} \|L_n(x)\|_2 \cdot \|L_n(x) + L_{n+1}(x)\|_2 \\
& \geq C \cdot \|u(x)\|_{{}_{-1}I_x^{\frac{\alpha_1-1}{2}} [L_2(-1,1)]} \cdot \|v(x)\|_{{}_xI_1^{\frac{\alpha_1+1}{2}} [L_2(-1,1)]}.
\end{aligned}$$

□

By [2], there is a unique solution to problem (3.45).

**Theorem 3.22.** *If  $|d| < \frac{\Gamma^2(\frac{\alpha_1-\alpha_2}{2}+1)}{2^{\alpha_1-\alpha_2+1}}$ , then the variation formulation (3.45) admits a unique solution  $u(x) \in X(-1, 1)$  satisfying*

$$(3.55) \quad \|u(x)\|_2 \leq C \|u(x)\|_q \leq C \|u(x)\|_{I^{\frac{\alpha_1-1}{2}} [L_2(-1,1)]} \leq C \|f\|_{W^{-1,p}(-1,1)},$$

where  $p = \frac{2}{\alpha_1}$ ,  $q = \frac{2}{2-\alpha_1}$ .

3.2.2. *Petrov-Galerkin tau methods.* Define

$$(3.56) \quad \Phi_N[-1, 1] := \left\{ \phi_N(x) : e^{\lambda x} \phi_N(x) \in {}_{-1}I_x^{\frac{\alpha_1-1}{2}} [P_{N-1}[-1, 1]], \text{ and } {}_{-1}D_x^{\alpha_1-1} (\phi_N(x))|_{x=-1} = 0 \right\},$$

and

$$(3.57) \quad \Psi_{N-1}[-1, 1] := e^{\lambda x} \cdot {}_xI_1^{\frac{\alpha_1+1}{2}} [P_{N-2}[-1, 1]].$$

It is clear from Corollary 2.13 that  $\Phi_N[-1, 1] \subseteq {}_{-1}I_x^{\frac{\alpha_1-1}{2}} [L_2(-1, 1)]$ , and  $\Psi_{N-1}[-1, 1] \subseteq {}_xI_1^{\frac{\alpha_1+1}{2}} [L_2(-1, 1)]$ .

To deal with the boundary condition on the right end-point, we choose Petrov-Galerkin tau method to approximation (3.45). That is, find  $u_N(x) \in \Phi_N[-1, 1]$ , such that  ${}_{-1}D_x^{\alpha_1-1} (u_N(x))|_{x=1} = u_b$ , and

$$(3.58) \quad B_2(u_N, \psi_{N-1}) = \langle f, \psi_{N-1} \rangle \quad \forall \psi_{N-1}(x) \in \Psi_{N-1}[-1, 1].$$

Similar to Lemmas 3.20 and 3.21, we can prove that  $B_2(\cdot, \cdot)$  satisfies the discrete inf-sup condition as well as the discrete ‘‘transposed’’ inf-sup condition. The following theorem holds.

**Theorem 3.23.** *If  $|d| < \frac{\Gamma^2(\frac{\alpha_1-\alpha_2}{2}+1)}{2^{\alpha_1-\alpha_2+1}}$ , then the discrete problem (3.58) admits a unique solution  $u_N(x) \in \Phi_N[-1, 1]$ , satisfying*

$$(3.59) \quad \|u_N(x)\|_{I^{\frac{\alpha_1-1}{2}} [L_2(-1,1)]} \leq C \|f(x)\|_{W^{-1,p}(-1,1)}.$$

Moreover, if  $u(x)$  is the solution of (3.45), we have

$$(3.60) \quad \|u - u_N\|_2 \leq C \|u - u_N\|_q \leq C \|u - u_N\|_{I^{\frac{\alpha_1-1}{2}} (L_2)} \leq C \inf_{\phi_N \in \Phi_N[-1,1]} \|u - \phi_N\|_{I^{\frac{\alpha_1-1}{2}} (L_2)},$$

where  $p = \frac{2}{\alpha_1}$ ,  $q = \frac{2}{2-\alpha_1}$ .

3.2.3. *Error estimates.*

**Lemma 3.24.** *If  ${}_aD_x^{\alpha_1, \lambda} \tilde{u}(x) + d \cdot {}_aD_x^{\alpha_2, \lambda} \tilde{u}(x) = \tilde{f}(x)$ , with  $1 < \alpha_2 < \alpha_1 < 2$ , then the following descriptions are equivalent:*

(i)

$$(3.61) \quad e^{\lambda x} \tilde{u}(x) \in {}_{-1}I_x^{\frac{\alpha_1-1}{2}} [H^m(-1, 1)], \quad m \geq 0;$$

(ii)

$$(3.62) \quad \tilde{u}(x) = e^{-\lambda x} (\tilde{u}_1(x) + \tilde{u}_2(x)),$$

where

$$(3.63a) \quad \tilde{u}_1(x) \in (1+x)^{\frac{\alpha_1-1}{2}} \cdot P_{m-1}[-1, 1],$$

(3.63b)

$$\tilde{u}_2(x) \in W^{m,q}(-1, 1), \quad q = \frac{2}{2-\alpha_1}, \quad \text{and } \tilde{u}_2(x) = o\left((1+x)^{m-1+\frac{\alpha_1}{2}}\right), \quad \text{as } x \rightarrow -1;$$

(iii)

$$(3.64) \quad \tilde{f}(x) = e^{-\lambda x} (\tilde{f}_1(x) + \tilde{f}_2(x)),$$

where

$$(3.65a) \quad \tilde{f}_1(x) \in (1+x)^{-(1+\alpha_1)/2} \cdot P_{m-1}[-1, 1] + (1+x)^{(\alpha_1-1)/2-\alpha_2} \cdot P_{m-1}[-1, 1],$$

(3.65b)

$$\tilde{f}_2(x) \in W^{m-1,p}(-1, 1), \quad p = \frac{2}{\alpha_1}, \quad \text{and } \tilde{f}_2(x) = o\left((1+x)^{m-1-\frac{\alpha_1}{2}}\right), \quad \text{as } x \rightarrow -1.$$

*Remark 3.25.* Here we note that from (3.63a)-(3.63b), it can be seen  ${}_{-1}D_x^{\alpha-2, \lambda} \tilde{u}(x)|_{x=-1} = 0$  holds.

*Proof.* It is easy to see that

$$(3.66) \quad e^{\lambda x} \tilde{f}(x) = D({}_{-1}D_x^{\alpha_1-1} (e^{\lambda x} \tilde{u}(x)) + d \cdot {}_{-1}D_x^{\alpha_2-1} (e^{\lambda x} \tilde{u}(x))) := Dg(x).$$

Denote  $\beta := \alpha_1 - 1$ . It is also clear that  $g(x) := g_1(x) + g_2(x)$ , where

$$(3.67a) \quad g_1(x) \in (1+x)^{-\beta/2} \cdot P_{m-1}[-1, 1] + (1+x)^{\beta/2-\alpha_2} \cdot P_{m-1}[-1, 1],$$

(3.67b)

$$g_2(x) = o\left((1+x)^{m-\frac{\beta}{2}-\frac{1}{2}}\right), \quad \text{as } x \rightarrow -1, \quad \text{and } g_2(x) \in W^{m,p}(-1, 1), \quad p = \frac{2}{1+\beta},$$

is equivalent to (iii) of Lemma 3.24. Therefore, the proof can be completed by Theorem 3.9.  $\square$

Let

$$(3.68) \quad P_N^0[-1, 1] = \{p : p(x) \in P_N[-1, 1], \text{ and } p(-1) = 0\}.$$

The following result can be derived from Lemma 3.24.

**Theorem 3.26.** *Let  $u(x)$  solve (3.1) with  $1 < \alpha_2 < \alpha_1 < 2$ . Then the following descriptions are equivalent to each other:*

(i)

$$(3.69) \quad e^{\lambda x} u(x) \in X(-1, 1);$$

(ii)

$$(3.70) \quad u(x) = e^{-\lambda x} (u_1(x) + u_2(x)),$$

where

$$(3.71a) \quad u_1(x) \in (1+x)^{\frac{\alpha_1-1}{2}} \cdot P_{m-1}^0[-1, 1],$$

(3.71b)

$$u_2(x) \in W^{m,q}(-1, 1), \text{ and } u_2(x) = \begin{cases} o\left((1+x)^{m-1+\frac{\alpha_1}{2}}\right), & \text{if } m \geq 1, \\ o\left((1+x)^{-1+\alpha_1}\right), & \text{if } m = 0, \end{cases} \text{ as } x \rightarrow -1;$$

(iii)

$$(3.72) \quad f(x) = e^{-\lambda x} (f_1(x) + f_2(x)),$$

where

$$(3.73a) \quad f_1(x) \in (1+x)^{-(1+\alpha_1)/2} \cdot P_{m-1}^0[-1, 1] + (1+x)^{(\alpha_1-1)/2-\alpha_2} \cdot P_{m-1}^0[-1, 1],$$

(3.73b)

$$f_2(x) \in W^{m-1,p}(-1, 1), \text{ and } f_2(x) = \begin{cases} o\left((1+x)^{m-1-\frac{\alpha_1}{2}}\right), & \text{if } m \geq 1, \\ o\left((1+x)^{-1}\right), & \text{if } m = 0, \end{cases} \text{ as } x \rightarrow -1,$$

where  $q = \frac{2}{2-\alpha_1}$ ,  $p = \frac{2}{\alpha_1}$ .

Combining Lemma 3.8 and Theorem 3.26, we can get the error estimate.

**Theorem 3.27.** Let  $e^{\lambda x} u(x) \in {}_{-1}I_x^{\frac{\alpha_1-1}{2}} [H^m(-1, 1)]$ ,  $m \geq 0$ , solve (3.45), and  $u_N(x)$  solve (3.58). If  $|d| < \frac{\Gamma(\alpha_1-\alpha_2+1)}{2^{\alpha_1-\alpha_2}}$ , then there is a constant  $C$  such that

$$(3.74) \quad \|u(x) - u_N(x)\|_2 \leq CN^{-m} \|f_2(x)\|_{W^{m-1,p}(-1,1)},$$

where  $f_2(x)$  is described in (3.73b).

*Proof.* If  $e^{\lambda x} u(x) \in {}_{-1}I_x^{\frac{\alpha_1-1}{2}} [H^m(-1, 1)]$ , then by the above analysis we can get

$$(3.75) \quad \begin{aligned} & D_{-1}^m D_x^{\frac{\alpha_1-1}{2}} (e^{\lambda x} u(x)) \\ &= {}_{-1}I_x^{\frac{\alpha_1-1}{2}} D_{-1}^m D_x^{\alpha_1-1} (e^{\lambda x} u(x)) \\ &= {}_{-1}I_x^{\frac{\alpha_1-1}{2}} D^{m-1} (e^{\lambda x} f(x)) - d \cdot {}_{-1}I_x^{\alpha_1-\alpha_2} D_{-1}^m D_x^{\frac{\alpha_1-1}{2}} (e^{\lambda x} u(x)), \end{aligned}$$

which implies by Lemma 2.9 that

$$(3.76) \quad \begin{aligned} & \left\| D_{-1}^m D_x^{\frac{\alpha_1-1}{2}} (e^{\lambda x} u(x)) \right\|_2 \\ & \leq C \|D^{m-1} f_2(x)\|_{L_p(-1,1)} + |d| \cdot \frac{2^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \left\| D_{-1}^m D_x^{\frac{\alpha_1-1}{2}} (e^{\lambda x} u(x)) \right\|_2. \end{aligned}$$

Thus,

$$(3.77) \quad \left(1 - |d| \cdot \frac{2^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right) \left\| D_{-1}^m D_x^{\frac{\alpha_1-1}{2}} u(x) \right\|_2 \leq C \|f_2(x)\|_{W^{m-1,p}(-1,1)}.$$

□

*Remark 3.28.* We note that although the limitation for the parameter  $d$  is required for the theoretical analysis, the examples in Section 4.3 suggest that it can be relaxed.

3.2.4. *Numerical implementation.* Here we construct trial functions and test functions, respectively, as:

$$(3.78) \quad \phi_n(x) := e^{-\lambda x} {}_{-1}I_x^{\frac{\alpha_1-1}{2}} (L_n(x) + L_{n+1}(x)),$$

for  $0 \leq n \leq N-1$ , and

$$(3.79) \quad \psi_n(x) := e^{\lambda x} {}_xI_1^{\frac{\alpha_1+1}{2}} L_n(x) = \frac{\Gamma(n+1)}{\Gamma(n+1 + \frac{\alpha_1+1}{2})} e^{\lambda x} (1-x)^{\frac{\alpha_1+1}{2}} J_n^{\frac{\alpha_1+1}{2}, -\frac{\alpha_1+1}{2}}(x),$$

for  $0 \leq n \leq N-2$ . Denote

$$(3.80) \quad u_N(x) = \sum_{n=0}^{N-1} \hat{u}_n \phi_n(x) = e^{-\lambda x} \sum_{n=0}^{N-1} \hat{u}_n \frac{\Gamma(n+2)}{\Gamma(n+2 + \frac{\alpha_1-1}{2})} (1+x)^{\frac{1+\alpha_1}{2}} J_n^{\frac{1-\alpha_1}{2}, \frac{1+\alpha_1}{2}}(x).$$

By Eqs. (3.46), (3.47), (3.32), (3.33), (3.34), and (3.37), a tau method for problem (3.1) with  $1 \leq \alpha_2 < \alpha_1 < 2$  can be written as

$$(3.81) \quad \mathbf{B}\mathbf{u} = \mathbf{f},$$

where  $\mathbf{u} = [\hat{u}_0, \hat{u}_1, \dots, \hat{u}_{N-1}]^T$ ,  $\mathbf{f}_k = \int_{-1}^1 f(x) \psi_k(x) dx$ ,  $k = 0, 1, \dots, N-2$ ,  $\mathbf{f}_N = 2^{\frac{\alpha_1-3}{2}} e^{\lambda} u_b$ ,  $\mathbf{B} = \mathbf{B}_1 + d \cdot \mathbf{B}_2$ , and  $\mathbf{B}_1$  is the same as in (3.40),

$$(3.82) \quad (\mathbf{B}_2)_{k,n} = \frac{\Gamma(k+2)\Gamma(n+1)}{\Gamma(k+2 + \frac{\alpha_1-\alpha_2}{2})\Gamma(n+1 + \frac{\alpha_1-\alpha_2}{2})} \cdot \int_{-1}^1 (1-x)^{\frac{\alpha_1-\alpha_2}{2}} (1+x)^{\frac{\alpha_1-\alpha_2}{2}+1} J_n^{-\frac{\alpha_1-\alpha_2}{2}, \frac{\alpha_1-\alpha_2}{2}+1}(x) J_k^{\frac{\alpha_1-\alpha_2}{2}, -\frac{\alpha_1-\alpha_2}{2}}(x) dx,$$

for  $k = 0, 1, \dots, N-2$ ,  $n = 0, 1, \dots, N-1$ .

$$(3.82) \quad (\mathbf{B}_2)_{N-1,n} = \frac{(n+1)\Gamma(n+1 + \frac{\alpha_1-1}{2})}{\Gamma(n+2 - \frac{\alpha_1-1}{2})\Gamma(\frac{\alpha_1+1}{2})}, \quad n = 0, 1, \dots, N-1.$$

#### 4. NUMERICAL TESTS

In what follows, we provide some numerical results to support the analysis of the proposed schemes and to validate our error analysis. In all following tests, we take  $\lambda = 1$ .

##### 4.1. Tempered fractional advection with $d = 0$ .

**Example 4.1.** We first consider (3.1) with the exact solution as

$$(4.1) \quad u(x) = e^{-x} \cdot {}_{-1}I_x^{\frac{\alpha_1}{2}} v(x) = \begin{cases} -\frac{(x+1)^{\frac{\alpha_1}{2}}}{\Gamma(1+\frac{\alpha_1}{2})}, & x \in [-1, 0], \\ \frac{2x^{\frac{\alpha_1}{2}} - (x+1)^{\frac{\alpha_1}{2}}}{\Gamma(1+\frac{\alpha_1}{2})}, & x \in (0, 1], \end{cases}$$

and

$$(4.2) \quad f(x) = \begin{cases} -\frac{(x+1)^{-\frac{\alpha_1}{2}}}{\Gamma(1-\frac{\alpha_1}{2})} \cdot e^{-x}, & x \in [-1, 0], \\ \frac{2x^{-\frac{\alpha_1}{2}} - (x+1)^{-\frac{\alpha_1}{2}}}{\Gamma(1-\frac{\alpha_1}{2})} \cdot e^{-x}, & x \in (0, 1]. \end{cases}$$

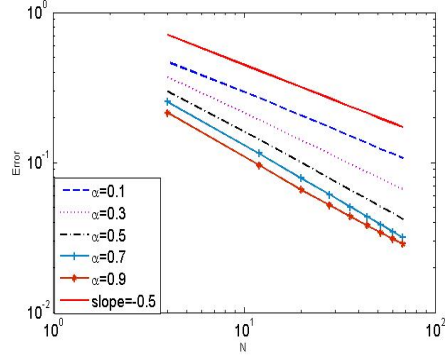


FIGURE 1. The numerical  $L_2$  errors for different  $\alpha_1$  of problem (3.1) with real solution as (4.1).

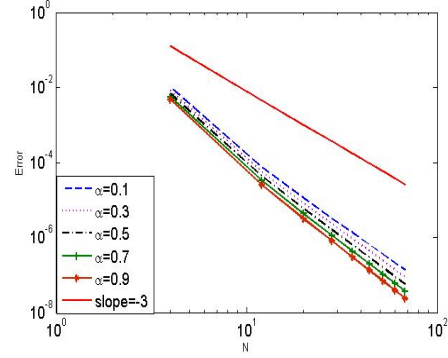


FIGURE 2. The numerical  $L_2$  errors for different  $\alpha_1$  of problem (3.1) with real solution as (4.4).

Here

$$(4.3) \quad v(x) = \begin{cases} -1, & x \in [-1, 0], \\ 1, & x \in (0, 1], \end{cases}$$

belongs to  $L_2(-1, 1)$ .

Fig. 1 verifies that as expected in Remark 3.16, errors in the sense of  $L_2(-1, 1)$  decay, even at the rate of  $N^{-1/2}$ . We can see that for larger  $\alpha_1$ , the error is smaller.

**Example 4.2.** Next, we consider (3.1) with the exact solution as

$$(4.4) \quad u(x) = e^{-x} \cdot {}_{-1}I_x^{\frac{\alpha_1}{2}} v(x) = \begin{cases} -\frac{(x+1)^{3+\frac{\alpha_1}{2}}}{\Gamma(4+\frac{\alpha_1}{2})}, & x \in [-1, 0], \\ \frac{2x^{3+\frac{\alpha_1}{2}} - (x+1)^{3+\frac{\alpha_1}{2}}}{\Gamma(4+\frac{\alpha_1}{2})}, & x \in (0, 1], \end{cases}$$

and

$$(4.5) \quad f(x) = \begin{cases} -\frac{(x+1)^{3-\frac{\alpha_1}{2}}}{\Gamma(4-\frac{\alpha_1}{2})} \cdot e^{-x}, & x \in [-1, 0], \\ \frac{2x^{3-\frac{\alpha_1}{2}} - (x+1)^{3-\frac{\alpha_1}{2}}}{\Gamma(4-\frac{\alpha_1}{2})} \cdot e^{-x} & x \in (0, 1], \end{cases}$$

where

$$(4.6) \quad D^{(3)}v(x) = \begin{cases} -1, & x \in [-1, 0], \\ 1, & x \in (0, 1]. \end{cases}$$

Thus  $v(x)$  belongs to  $H^3(-1, 1)$ .

We can see from Fig. 2 that the errors are in agreement with the estimate in Theorem 3.12.

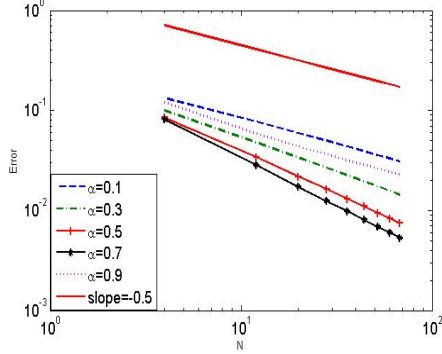


FIGURE 3. The numerical  $L_2$  errors for different  $\alpha_1$  of problem (3.1) with the right function as (4.7).

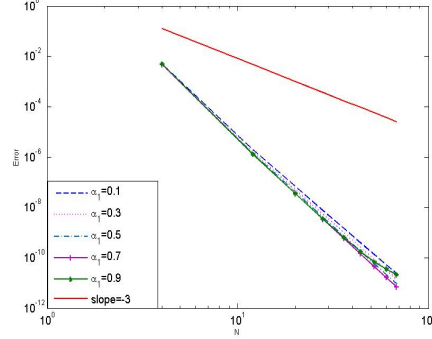


FIGURE 4. The numerical  $L_2$  errors for different  $\alpha_1$  of problem (3.1) with  $\alpha_2 = 0$ ,  $d = -500$ , and real solution as (4.8), where  $m = 3$ ,  $\gamma = 0.3$ .

**Example 4.3.** Thirdly, we consider (3.1) with

$$(4.7) \quad f(x) = (1+x)^{-\frac{\alpha_1}{2}-0.3},$$

so  $f_2(x) = e^x \cdot f(x) = o\left((1+x)^{-\frac{\alpha_1}{2}-\frac{1}{2}}\right)$  as  $x \rightarrow -1$ , and  $f_2(x) \in L_p(-1, 1)$ , where  $q = \frac{2}{1+\alpha_1}$ .

Since the explicit form of the correspondingly exact solution is not available, we numerically compute it by using a Gauss quadrature method with a large number of nodes. The convergence behaviours for different  $\alpha$  are depicted in Fig. 3.

#### 4.2. Tempered fractional advection with $d \neq 0$ .

**Example 4.4.** Now we consider the problem (3.1) with  $0 \leq \alpha_2 < \alpha_1 < 1$ , and  $d \neq 0$ . We discuss a real solution of the form

$$(4.8) \quad u(x) = e^{-x} \cdot (1+x)^{m+\frac{\alpha_1}{2}-\gamma},$$

with

$$(4.9) \quad f(x) = e^{-x} \cdot \frac{\Gamma(m+1+\frac{\alpha_1}{2}-\gamma)}{m+1-\frac{\alpha_1}{2}-\gamma} (1+x)^{m-\frac{\alpha_1}{2}-\gamma}.$$

Take  $m = 3$ , and  $\gamma = 0.3$ . Thus,  $u_2(x) = e^x \cdot u(x) \in W^{3,q}(-1, 1)$ , where  $q = \frac{2}{1-\alpha}$ , and  $u_2(x) = o\left((1+x)^{3+\frac{\alpha_1}{2}-\frac{1}{2}}\right)$  as  $x \rightarrow -1$ . Therefore, by Theorem 3.12, at least third order of convergence is expected for all different  $\alpha_1$  and  $\alpha_2$ . Fig. 4 shows the convergence behaviours for all values of  $\alpha_1$ , with  $\alpha_2 = 0$  and  $d = -500$ , while Fig. 5 demonstrates the convergence behaviours for different  $\alpha_1$ , with  $\alpha_2 = 0$  and  $d = 500$ , from which we can see that in applications, the parameter  $d$  does not appeared to be as limited as in the theoretical analysis.

If we take  $m = 0$  and  $\gamma = 0.3$ , then  $u(-1)$  might be non-zero or even infinite. We can also see even  $N^{-1/2}$  of convergence in Fig. 6 with  $d = 5$ .



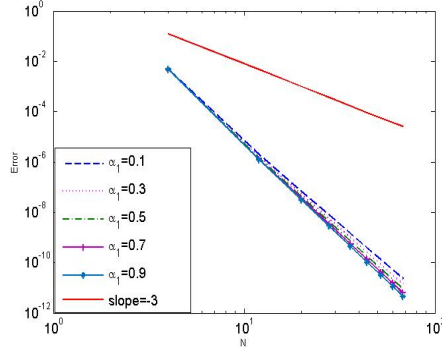


FIGURE 5. The numerical  $L_2$  errors for different  $\alpha_1$  of problem (3.1) with  $\alpha_2 = 0$ ,  $d = 500$ , and real solution as (4.8), where  $m = 3$ ,  $\gamma = 0.3$ .

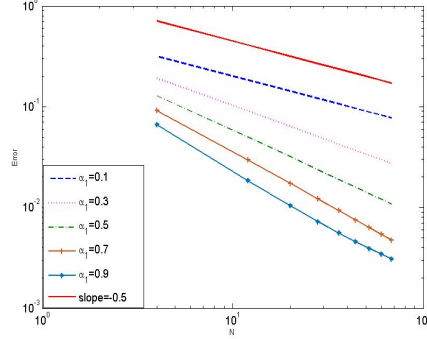


FIGURE 6. The numerical  $L_2$  errors for different  $\alpha_1$  of problem (3.1) with  $\alpha_2 = 0$ ,  $d = 5$ , and real solution as (4.8), where  $m = 0$ ,  $\gamma = 0.3$ .

#### 4.3. Tempered fractional diffusion.

**Example 4.5.** Let us finally consider (3.1) with  $1 \leq \alpha_2 < \alpha_1 < 2$ , and consider the solution of the form

$$(4.10) \quad u(x) = e^{-x} \cdot (1+x)^{\beta-1} E_{\gamma, \beta}((x+1)^\gamma),$$

with

$$(4.11) \quad f(x) = e^{-x} \cdot [(1+x)^{\beta-1-\alpha_1} E_{\gamma, \beta-\alpha_1}((x+1)^\gamma) + d \cdot (1+x)^{\beta-1-\alpha_2} E_{\gamma, \beta-\alpha_2}((x+1)^\gamma)].$$

We take  $d = -1$ ,  $\beta = 4$ , and  $\gamma = 1$ . We can see that although  $u_2(x) = e^x \cdot u(x) \in W^{m,q}(-1, 1) \forall m \in \mathbb{N}$ ,  $q = \frac{2}{2-\alpha}$ ,  $u_2(x) = o((1+x)^{3-1+\frac{\alpha_1}{2}})$  as  $x \rightarrow -1$ . Therefore, by Theorem 3.27, at least third order of convergence is expected for all values of  $\alpha_1$  and  $\alpha_2$ . Fig. 7 shows the convergence for values of different  $\alpha_1$  with  $\alpha_2 = 1$ , while Fig. 8 demonstrates the convergence for different  $\alpha_2$  with  $\alpha_1 = 1.99$ , from which it can be seen that the numerical errors mainly depend on  $\alpha_1$ .

Next, we take  $d = 100$ ,  $\beta = \frac{\alpha_1+1}{2} + 1$ , and  $\gamma = 1$ . We expect the errors decay exponentially, which can be seen in Figs. 9 and 10.

## 5. CONCLUSION

In this paper, we first introduce fractional integral spaces on a finite domain. Functions in this space are not required to be zero at the boundary point when  $\alpha < 1$ , where  $\alpha$  is the space order. We show that this tempered fractional operators turn out equal to the standard Riemann-Liouville operators in the sense of the integral spaces. Based on this fact, we derive the variational formulation for the tempered fractional advection/diffusion problems in the weak sense and solve them using spectral methods. And we develop their error estimates for the solutions. We consider multi-term tempered fractional problems in a similar way. Through examples we confirm that the behavior of the highest order of the fractional derivative

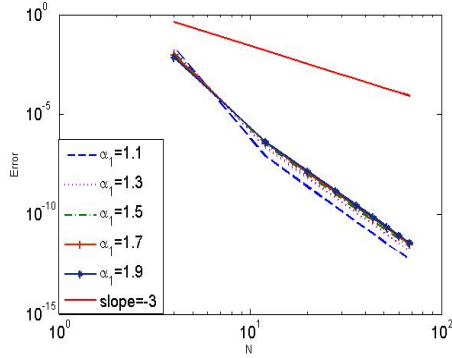


FIGURE 7. The numerical  $L_2$  errors for different  $\alpha_1$  of problem (3.1) with  $\alpha_2 = 1$ ,  $d = -1$ , and real solution as (4.10), where  $\beta = 4$ ,  $\gamma = 1$ .

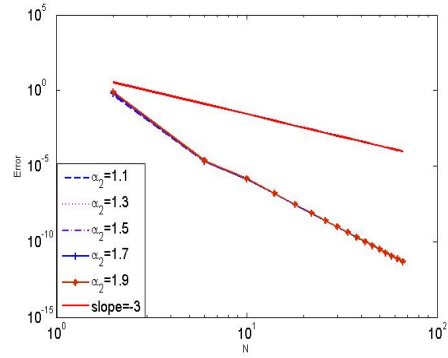


FIGURE 8. The numerical  $L_2$  errors for different  $\alpha_2$  of problem (3.1) with  $\alpha_1 = 1.99$ ,  $d = -1$ , and real solution as (4.10), where  $\beta = 4$ ,  $\gamma = 1$ .

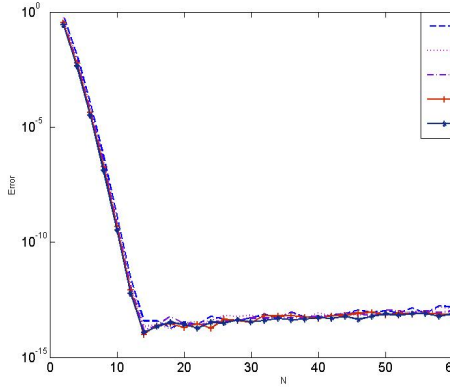


FIGURE 9. The numerical  $L_2$  errors for different  $\alpha_1$  of problem (3.1) with  $\alpha_2 = 1$ ,  $d = 100$ , and real solution as (4.10), where  $\beta = \frac{\alpha_1+1}{2} + 1$ ,  $\gamma = 1$ .

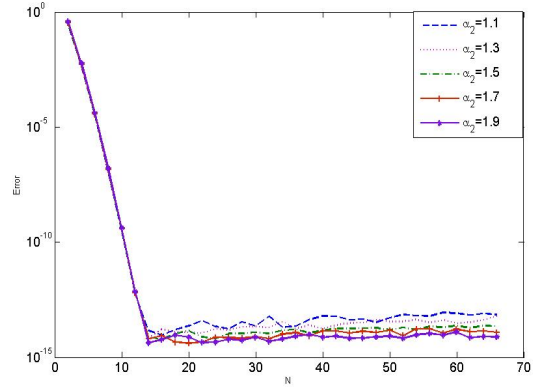


FIGURE 10. The numerical  $L_2$  errors for different  $\alpha_2$  of problem (3.1) with  $\alpha_1 = 1.99$ ,  $d = 100$ , and real solution as (4.10), where  $\beta = \frac{\alpha_1+1}{2} + 1$ ,  $\gamma = 1$ .

dominates the convergence in agreement with our past work. In future work, we shall extend this work to solve tempered fractional advection-diffusion problems.

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## APPENDIX A. APPENDIX

**Lemma A.1.** *If  $f(x) \in {}_a I_x^\alpha[L_1(a, b)]$ ,  $1 \leq n-1 \leq \alpha < n$ , then the condition (2.16) is equivalent to (2.18).*

*Proof.* If  $f(x) \in {}_a I_x^\alpha[L_1(a, b)]$ , then by Lemma 2.4 and the formula (2.115) of [25], there is

$$(A.1) \quad D^k f(x) = {}_a I_x^{\alpha-k} {}_a D_x^\alpha f(x) + \sum_{j=1}^n \frac{(x-a)^{\alpha-k-j}}{\Gamma(1+\alpha-k-j)} \cdot {}_a D_x^{\alpha-j} f(x)|_{x=a} = {}_a I_x^{\alpha-k} {}_a D_x^\alpha f(x),$$

for  $k = 1, 2, \dots, n-1$ . So, by Lemma 2.7 and Lemma 2.9, we have

$$(A.2) \quad {}_a D_x^\alpha f(x) \in L_1(a, b),$$

and thus

$$(A.3) \quad D^k f(x) \in L_1(a, b).$$

Therefore,

$$(A.4) \quad {}_a I_x^{n-\alpha} D^k f(x) = {}_a D_x^{\alpha-n+k} f(x) - \sum_{j=0}^{k-1} \frac{D^j f(a)}{\Gamma(1+n-\alpha-k+j)} (x-a)^{n-\alpha-k+j}.$$

On one hand, if

$$(A.5) \quad D^j f(a) = 0, \quad j = 0, 1, \dots, k-1,$$

are fulfilled, then putting  $x \rightarrow a$  in (A.4) we immediately obtain

$$(A.6) \quad {}_a D_x^{\alpha-n+k} f(x)|_{x=a} = 0.$$

One the other hand, if (A.6) is fulfilled, then multiplying both sides of (A.4) subsequently by  $(x-a)^{\alpha+k-n-j}$ , ( $j = k-1, k-2, \dots, 0$ ), and taking the limits as  $x \rightarrow a$ , we obtain  $D^{k-1} f(a) = 0, D^{k-2} f(a) = 0, \dots, Df(a) = 0, f(a) = 0$ , i.e., the condition (A.5).

Therefore, (2.16) is equivalent to (2.18).  $\square$

**Lemma A.2.** *Let  $n-1 \leq \alpha < n$ . If  $f(x) \in AC^n[a, b]$ , and  $D^j f(a) = 0$ ,  $j = 0, 1, \dots, n-2$ , then  $f(x) \in {}_a I_x^\alpha[L_1(a, b)]$ .*

*Proof.* Since  $D^j f(a) = 0$ ,  $j = 0, 1, \dots, n-2$ , by Lemma A.1, we can have

$$(A.7) \quad {}_a D_x^{\alpha-k} f(x)|_{x=a} = 0, \quad k = 1, 2, \dots, n-1.$$

By (2.9) and (2.10), there is  $D^n f(x) \in L_1(a, b)$  from  $f(x) \in AC^n[a, b]$ . So,

$$(A.8) \quad {}_a I_x^{n-\alpha} f(x)|_{x=a} = 0.$$

By (2.22), we can get

$$(A.9) \quad {}_a D_x^\alpha f(x) = {}_a I_x^{n-\alpha} D^n f(x) + \frac{D^{n-1} f(a)}{\Gamma(n-\alpha)} (x-a)^{n-1-\alpha} \in L_1(a, b).$$

Thus, from the formula (2.115) in [25] and Lemma 2.9, it can be yield that

$$(A.10) \quad {}_a D_x^{\alpha-k} f(x) = {}_a I_x^k {}_a D_x^\alpha f(x) \in L_1(a, b), \text{ for } k = 1, 2, \dots, n.$$

So,

$$(A.11) \quad {}_a I_x^{n-\alpha} f(x) \in AC^n[a, b].$$

Finally, from Lemma 2.4 we can get  $f(x) \in {}_a I_x^\alpha [L_1(a, b)]$ . □

## APPENDIX B. APPENDIX

Before proving Theorem 3.9, we recall a general Sobolev inequality ([13], p.270).

**Lemma B.1.** *Assume  $\varphi(x) \in W^{m,\gamma}(-1,1)$ ,  $m \geq 1$ ,  $\gamma > 1$ . Then  $\varphi(x) \in C^{m-1,1-\frac{1}{\gamma}}[-1,1]$ . Furthermore, we have the estimate*

$$(B.1) \quad \|\varphi(x)\|_{C^{m-1,1-\frac{1}{\gamma}}[-1,1]} \leq C \|\varphi(x)\|_{W^{m,\gamma}(-1,1)},$$

where the constant  $C$  depends only on  $m$ , and the Hölder space  $C^{m-1,1-\frac{1}{\gamma}}[-1,1]$  is defined as in (2.32).

Proof of Theorem 3.9:

We only prove the result for  $m \geq 1$ , since when  $m = 0$ ,  $P_{-1}[-1,1] = \{0\}$  as we defined in Section 3.1.2, and the result is immediate.

(i)  $\Rightarrow$  (ii) :

If  $e^{\lambda x} u(x) \in {}_{-1}I_x^{\alpha_1/2}[H^m(-1,1)]$ , then, by Lemma 2.7, there exists  $v(x) = {}_{-1}D_x^{\alpha_1/2}(e^{\lambda x} u(x)) \in H^m(-1,1)$ , such that  $e^{\lambda x} u(x) = {}_{-1}I_x^{\alpha_1/2} v(x)$ . Also, there is

$$(B.2) \quad v(x) = {}_{-1}I_x^m D^m v(x) + [C_0 + C_1(1+x) + \dots + C_{m-1}(1+x)^{m-1}] := v_2(x) + v_1(x),$$

and

$$(B.3) \quad e^{\lambda x} u(x) = {}_{-1}I_x^{m+\alpha_1/2} D^m v_2(x) + (1+x)^{\frac{\alpha_1}{2}} p_{m-1}(x) := u_2(x) + u_1(x),$$

where  $p_{m-1}(x) \in P_{m-1}[-1,1]$ .

Since  $D^m v(x) \in L_2(-1,1)$ , by (iii) of Lemma 2.9, we have

$$(B.4) \quad v_2(x) = {}_{-1}I_x^m D^m v(x) \in C^{m-1, \frac{1}{2}}[-1,1], \text{ and } v_2(x) = o\left((1+x)^{m-\frac{1}{2}}\right), \text{ as } x \rightarrow -1,$$

thus,

$$u_2(x) = {}_{-1}I_x^{m+\alpha_1/2} D^m v(x) \in C^{m-1, \frac{1}{2} + \frac{\alpha_1}{2}}[-1,1],$$

and

$$u_1(x) = o\left((1+x)^{m+\frac{\alpha_1}{2}-\frac{1}{2}}\right), \text{ as } x \rightarrow -1.$$

From (B.4)-(B), we see that

$$(B.5) \quad D^k v_2(x)|_{x=-1} = 0, \quad k = 0, 1, \dots, m-1,$$

therefore,

$$(B.6) \quad D^j u_2(x) = D^j {}_{-1}I_x^{\alpha_1/2} v_2(x) = {}_{-1}I_x^{\alpha_1/2} (D^j v_2(x)), \quad j = 0, 1, \dots, m.$$

In addition, since  $D^j v_2(x) \in L_2(-1,1)$ , we can get from (ii) of Lemma 2.9 that

$$(B.7) \quad D^j u_2(x) \in L_q(-1,1), \quad j = 0, 1, \dots, m,$$

where  $q = \frac{2}{1-\alpha_1}$ . That is,  $u_2(x) \in W^{m,q}(-1, 1)$ .

(ii)  $\Rightarrow$  (i):

If  $u_2(x) \in W^{m,q}(-1, 1)$ ,  $q = \frac{2}{1-\alpha_1}$ , then by Lemma B.1, we have  $u_2(x) \in C^{m-1, \frac{1}{2} + \frac{\alpha_1}{2}}[-1, 1]$ . With (B), we obtain

$$(B.8) \quad D^j u_2(x)|_{x=-1} = 0, \quad j = 0, 1, \dots, m-1,$$

and

$$(B.9) \quad D^k v_2(x) = D^k {}_{-1}D_x^{\alpha_1/2} u_2(x) = {}_{-1}D_x^{k + \frac{\alpha_1}{2}} u_2(x) = {}_{-1}I_x^{1 - \frac{\alpha_1}{2}} D^{k+1} u(x),$$

$k = 0, 1, \dots, m-1$ . By (iv) of Lemma 2.9, it follows

$$(B.10) \quad D^k v_2(x) \in c^{0, \frac{1}{2}}[-1, 1], \quad k = 0, 1, \dots, m-1.$$

So, by the definition of  $c^{0, \frac{1}{2}}[-1, 1]$  in (2.33), we have

$$(B.11) \quad \begin{aligned} \int_{-1}^1 |D^m v_2(x)|^2 dx &= \int_{-1}^1 \lim_{y \rightarrow x} \left| \frac{D^{m-1} v_2(y) - D^{m-1} v_2(x)}{y-x} \right|^2 dx \\ &= \int_{-1}^1 \lim_{y \rightarrow x} \left| o((y-x)^{-\frac{1}{2}}) \right|^2 dx < \infty. \end{aligned}$$

Thus,  $v_2(x) \in H^m(-1, 1)$ .

(ii)  $\Rightarrow$  (iii) :

Similar to the above analysis, we get

$$(B.12) \quad \begin{aligned} D^k f_2(x) &= {}_{-1}D_x^{k+\alpha_1} u_2(x) + d \cdot {}_{-1}D_x^{k+\alpha_2} u_2(x) \\ &= {}_{-1}I_x^{1-\alpha_1} D^{k+1} u_2(x) + d \cdot {}_{-1}I_x^{1-\alpha_2} D^{k+1} u_2(x), \quad k = 0, 1, \dots, m-1, \end{aligned}$$

and

$$(B.13) \quad {}_{-1}I_x^{1-\alpha_1} D^{k+1} u_2(x) \in c^{0, \frac{1-\alpha_1}{2}}[-1, 1], \quad k = 0, 1, \dots, m-1,$$

$$(B.14) \quad {}_{-1}I_x^{1-\alpha_2} D^{k+1} u_2(x) \in c^{0, \frac{1+\alpha_1}{2} - \alpha_2}[-1, 1], \quad k = 0, 1, \dots, m-1.$$

From the definition of  $c^{m, \gamma}[a, b]$  in (2.33), we see  $c^{0, \frac{1+\alpha_1}{2} - \alpha_2}[-1, 1] \subseteq c^{0, \frac{1-\alpha_1}{2}}[-1, 1]$ . Hence,

$$(B.15) \quad D^k f_2(x) \in c^{0, \frac{1-\alpha_1}{2}}[-1, 1], \quad k = 0, 1, \dots, m-1,$$

and

$$(B.16) \quad \int_{-1}^1 |D^m f_2(x)|^p dx = \int_{-1}^1 \lim_{y \rightarrow x} \left| o((y-x)^{-\frac{1+\alpha_1}{2}}) \right|^{\frac{2}{1+\alpha_1}} dx < \infty.$$

Thus,  $f_2(x) \in W^{m,p}(-1, 1)$ .

(iii)  $\Rightarrow$  (ii) :

Since  ${}_{-1}D_x^{\alpha_1-1, \lambda} u(x)|_{x=-1} = 0$ , there is  ${}_{-1}D_x^{\alpha_2-1, \lambda} u(x)|_{x=-1} = 0$ , and

$$(B.17) \quad e^{\lambda x} u(x) + d \cdot {}_{-1}I_x^{\alpha_1 - \alpha_2} (e^{\lambda x} u(x)) = {}_{-1}I_x^{\alpha_1} (e^{\lambda x} f(x)).$$

By

$$(B.18) \quad D^k f_2(x)|_{x=-1} = 0, \quad k = 0, 1, \dots, m-1,$$

we get

$$(B.19) \quad D^j [u_2(x) + d \cdot {}_{-1}I_x^{\alpha_1 - \alpha_2} u_2(x)] = D^j {}_{-1}I_x^{\alpha_1} f_2(x) = {}_{-1}I_x^{\alpha_1} (D^j f_2(x)), \quad j = 0, 1, \dots, m.$$

In addition, since  $D^j f_2(x) \in L_p(-1, 1)$ ,  $1 < p = \frac{2}{1+\alpha_1} < 1/\alpha_1$ , we get from (ii) of Lemma 2.9 that

$$(B.20) \quad D^j [u_2(x) + d \cdot {}_{-1}I_x^{\alpha_1 - \alpha_2} u_2(x)] = o\left((1+x)^{\frac{\alpha_1-1}{2}}\right), \quad \text{as } x \rightarrow -1,$$

and

$$(B.21) \quad D^j [u_2(x) + d \cdot {}_{-1}I_x^{\alpha_1 - \alpha_2} u_2(x)] \in L_q(-1, 1), \quad j = 0, 1, \dots, m,$$

which is independent of  $d$ , where  $q = \frac{2}{1-\alpha_1}$ . So,  $u_2(x) = o\left((1+x)^{\frac{m+\alpha_1-1}{2}}\right)$ , and  $u_2(x) \in W^{m,q}(-1, 1)$ .

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