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Rational constants of monomial derivations. (English summary)
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Let $k(X)=k(x_1,\dots,x_n)$ be a field of rational functions over a field k of characteristic 0. A monomial derivation is a derivation $d\colon k(X)\to k(X)$ such that $d(x_i)=x_1^{\beta_{i1}}\cdots x_n^{\beta_{in}}$, $i=1,\dots,n$, where $\beta_{ij}\in\mathbb{Z}$. The paper characterizes the field of constants $k(X)^d$, which is the set of elements φ of $k(X)$ such that $d(\varphi)=0$.

This problem originates in the search of first integrals of ordinary differential equations. Even in the very simple case of linear ordinary differential equations, there exist rational first integrals (take for example $\dot x=x$, $\dot y=-2x+2y$ and $\dot z=2y$, then $\frac{2x-y}{x^2}$ is a rational first integral; i.e., it is constant along a solution curve). Rational integrating functions also appear in the context of total differentials and Pfaff's problem. They were initially investigated by Poincaré and Painlevé.

The existence of first integrals is closely linked to the existence of Darboux polynomials F associated with the given derivation d: F is a Darboux polynomial when $d(F)=\Lambda F$, with Λ being the associated cofactor. For polynomial differential equations, for example, Darboux polynomials are partial first integrals. In the present context, which uses the language of differential algebra, Darboux polynomials coincide with generators of principal differential ideals. In this paper, they help in going from the study of $k[x_1,\dots,x_n]^d$ (constants of the polynomial ring) to the study of $k(x_1,\dots,x_n)^d$ (constants of the field of rational functions).

Rather than dealing directly with d, it is sometimes more convenient to use the associated factorisable derivation δ defined as $\delta(x_i)=x_i\sum_{j=1}^n\alpha_{ij}x_j$, $i=1,\dots,n$, where $\alpha=[\alpha_{ij}]$ denotes the matrix $\beta-I_{n\times n}$, with $\beta=[\beta_{ij}]$.

Using the interplay between both of these derivations, d and δ, together with inferences based on combinatorics and contradiction arguments (e.g. degree compatibility of the Darboux polynomials), the authors propose—apart from general results involving n variables—a characterization of the field of rational constants of monomial derivations in three variables (i.e. $k(x,y,z)^d$). These results are obtained based on:

- the use of groups of invariants of specific automorphisms of $k(x_1,\dots,x_n)$. Special emphasis is put on the diagonal automorphism $\sigma_u(x_i)=u_ix_i$, $i=1,\dots,n$, where $u=(u_1,\dots,u_n)$ is a sequence of elements of k^*;
- the classification of Lotka-Volterra derivations previously derived in the work [J. Moulin Ollagnier, Qual. Theory Dyn. Syst. **2** (2001), no. 2, 307–358; MR1913289 (2003f:34014); correction, "Liouvillian integration of the Lotka-Volterra system", preprint, 2005; per bibl.]. The Lotka-Volterra derivation is defined as $D(x)=x(Cy+z)$, $D(y)=y(Az+x)$, $D(z)=z(Bx+y)$, with $A,B,C\in k$. Incidentally, this derivation arises from the differential equations describing predator-pray dynamics, which are complex and often of a chaotic nature.

The authors relate D to d and δ. For example, consider the monomial derivation $d(x) = xy^{p_2}z^{p_3}$, $d(y) = x^{q_1}yz^{q_3}$, $d(z) = x^{r_1}y^{r_2}z$, with p_2, p_3, q_1, q_3, r_1, r_2 all belonging to $\mathbb{Z}$ and fulfilling $p_2q_3r_1 + p_3q_1r_2 \neq 0$. The associated factorisable derivation δ together with the diagonal automorphism $\sigma(x) = q_1^{-1}x$, $\sigma(y) = r_2^{-1}y$, and $\sigma(z) = p_3^{-1}z$ then leads to a Lotka-Volterra derivation $D = \sigma\delta\sigma^{-1}$. Using similar relations between d, δ, and D, it is possible to recast the classification results concerning the Lotka-Volterra derivations D to corresponding properties of specific monomial derivations d.

This leads to consequences for other types of derivations as well, and in particular for the Jouanolou derivative $d_J(x) = y^s$, $d_J(y) = z^s$ and $d_J(z) = x^s$ with $s \in \mathbb{Z}$. The authors show that $k(x, y, z)^{d_J} \neq k$ if and only if $s \in \{-2, 0, 1\}$. The proof is purely algebraic, elegant, and short, notwithstanding that it is a consequence of the Lotka-Volterra classification of [op. cit.]. The result is a generalization to three variables of the existence of only trivial constants for $s \geq 2$, which was proved for an arbitrary number of variables by many authors. The corresponding proofs are, however, complicated and rely—apart from algebraic arguments—on algebraic geometry and dynamical system properties.

Another implication is a full list of all possible homogeneous monomial derivations of three variables having degree two, three and four, for which the field of constants is trivial. There are respectively 40 derivations of degree two (divided into 8 parts), 188 derivations of degree three (divided into 34 parts), and 538 derivations of degree four (separated into 91 parts).

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.