Null-controllability, exact controllability, and stabilization of hyperbolic systems for the optimal time

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Abstract—In this paper, we discuss our recent works on the null-controllability, the exact controllability, and the stabilization of linear hyperbolic systems in one dimensional space using boundary controls on one side for the optimal time. Under precise and generic assumptions on the boundary conditions on the other side, we first obtain the optimal time for the null and the exact controllability for these systems for a generic source term. We then prove the null-controllability and the exact controllability for any time greater than the optimal time and for any source term. Finally, for homogeneous systems, we design feedbacks which stabilize the systems and bring them to the zero state at the optimal time. Extensions for the non-linear homogeneous system are also discussed.

I. INTRODUCTION

Linear hyperbolic systems in one dimensional space are frequently used in modelling of many systems such as traffic flow, heat exchangers, and fluids in open channels. The stability and boundary stabilization of these hyperbolic systems have been studied intensively in the literature, see, e.g., [3] and the references therein. In this paper, we are interested in the null-controllability, the exact controllability, and the stabilization at finite time of linear hyperbolic systems in one dimensional space using boundary controls on one side. More precisely, we consider the system for

\[ \partial_t w(t,x) = \sum_{i=1}^{n} \partial_{x_i} w(t,x) + C(x)w(t,x) \]  

(1)

Here \( w = (w_1, \ldots, w_n)^T : \mathbb{R}_+ \times (0,1) \to \mathbb{R}^n \) (\( n \geq 2 \)), \( \Sigma \) and \( C \) are \( (n \times n) \) real matrix-valued functions defined in \([0,1]\). We assume that for every \( x \in [0,1] \), \( \Sigma(x) \) is diagonal with \( m \) 1 distinct positive eigenvalues, and \( k = n - m \) 1 distinct negative eigenvalues. Using Riemann coordinates, one might assume that \( \Sigma(x) \) is of the form

\[ \Sigma(x) = \text{diag}( -\lambda_1(x), \ldots, -\lambda_k(x), \lambda_{k+1}(x), \ldots, \lambda_n(x)), \]

(2)

where

\[ -\lambda_1(x) < \cdots < -\lambda_k(x) < 0 < \lambda_{k+1}(x) < \cdots < \lambda_n(x). \]

(3)

Throughout the paper, we assume that

\[ \lambda_i \text{ is Lipschitz on } [0,1] \text{ for } 1\leq i \leq n (= k + m). \]

(4)

We also assume that

\[ C \in (L^\infty([0,1]))^{n \times n}. \]

(5)

We are interested in the following type of boundary conditions and boundary controls. The boundary conditions at \( x = 0 \) are given by, for \( t \geq 0 \),

\[ (w_1, \ldots, w_k)^T(t,0) = B(w_{k+1}, \ldots, w_{k+m})^T(t,0) \]

(6)

for some \((k \times m)\) real constant matrix \( B \), and the boundary controls at \( x = 1 \) are, for \( t \geq 0 \),

\[ w_{k+1}(t,1) = W_{k+1}(t), \quad \ldots, \quad w_{k+m}(t,1) = W_{k+m}(t), \]

(7)

where \( W_{k+1}, \ldots, W_{k+m} \) are controls.

Let us recall that the control system (1), (6), and (7) is null-controllable (resp. exactly controllable) at the time \( T > 0 \) if, for every initial data \( w_0 : (0,1) \to \mathbb{R}^n \) in \( L^2(0,1)^n \) (resp. for every initial data \( w_0 : (0,1) \to \mathbb{R}^n \) in \( L^2(0,1)^n \)) and for every (final) state \( w_T : (0,1) \to \mathbb{R}^n \) in \( L^2(0,1)^n \), there is a control \( W = (W_{k+1}, \ldots, W_{k+m})^T : (0,T) \to \mathbb{R}^m \) such that the solution of (1), (6), and (7) satisfying \( w(t = 0,x) = w_0(x) \) vanishes (resp. reaches \( w_T \)) at the time \( T \); \( w(t = T,x) = 0 \) (resp. \( w(t = T,x) = w_T \)).

Similar definitions hold for \( w_0 \in L^2(0,1)^n \) (resp. \( w_0, w_T \in L^2(0,1)^n \)) with \( W \in L^2(0,T)^m \). Set

\[ \tau_i := \int_0^1 \frac{1}{\lambda_i(\xi)} d\xi \text{ for } 1 \leq i \leq n. \]

(8)

The exact controllability, the null-controllability, and the boundary stabilization problem of hyperbolic system in one dimension have been widely investigated in the literature for almost half a century. The pioneer works date back to Rauch and Taylor [28] and Russell [30]. In particular, it was shown, see [30, Theorem 3.2], that the system (1), (6), and (7) is null-controllable if the time

\[ T \geq \tau_k + \tau_{k+1}, \]

and is exact controllable if \( k = m \) and \( B \) is invertible.

The extension of this result for quasilinear system was initiated by Greenberg and Li [19] and Slemrod [31].

Concerning the stabilisation of (1), many articles are based on the boundary conditions with the following specific form

\[ \begin{pmatrix} w_-(t,0) \\ w_+(t,1) \end{pmatrix} = G \begin{pmatrix} w_-(t,0) \\ w_+(t,1) \end{pmatrix}, \]

(9)

where \( G : \mathbb{R}^n \to \mathbb{R}^n \) is a suitable smooth vector field. Three approaches have been proposed to deal with (9). The first one is based on the characteristic method. This method was previously investigated in Greenberg and Li [19] for \( 2 \times 2 \) systems and Qin [27] (see also Li [26]) for a generalization to \( n \times n \) homogeneous nonlinear hyperbolic systems in the framework of \( C^1 \)-norm. The second approach is based on
Lyapunov functions see [7], [5] (see also [25], [6]). The third approach [11] is based on the study of delay equations works for $W^{2,p}$-norm with $p \geq 1$. These works typically impose restrictions on the magnitude of the coupling coefficients.

This restriction was overcome via backstepping approach. This was first proposed by Coron et al. [17] for $2 \times 2$ system ($m = k = 1$). Later this approach has been extended and now can be applied for general pairs $(m, k)$, see [18], [21], [1], [9]. In [17], the null-controllability is achieved via a feedback law for the time $\tau_1 + \tau_2$ with $m = k = 1$ via backstepping approach. In [21], the authors considered the case where $\Sigma$ is constant and obtained feedback laws for the null-controllability at the time $T_{opt}$.

It was later showed in [1], [9] that one can reach the null-controllability at the time $T_{2} := \tau_k + \tau_{k+1}$. (11)

Set

$$T_{opt} := \begin{cases} \max \{\tau_1 + \tau_{m+1}, \ldots, \tau_k + \tau_{m+k}\}; & \text{if } m \geq k, \\ \max \{\tau_{k+1-m} + \tau_{k+1}, \tau_{k+2-m} + \tau_{k+2}, \ldots, \tau_k + \tau_{m}\}; & \text{if } m < k. \end{cases}$$

(12)

In this paper, we report our recent works [13], [14], [15], [16] on the null-controllability, the exact controllability, and the stabilization of (1), (6), and (7). We show that the null-controllability holds at $T_{opt}$ for generic $B$ and $C$ and the null-controllability holds for any $T > T_{opt}$ under a precise condition on $B$ ($B \in \mathcal{B}$ given in (13)), which holds for almost every matrix $k \times m$ matrix $B$. Similar conclusions holds for the exact controllability (with $B \in \mathcal{B}_e$ given in (14)) under natural, additional condition $m \geq k$. When the system is homogeneous, we show that the null-controllability is achieved via a time-independent feedback and there are Lyapunov’s functions associated with these feedbacks. This result also holds for quasilinear setting. The starting point of our approach in the inhomogeneous case is the backstepping approach.

Remark 1.1: The backstepping approach for the control of partial differential equations was pioneered by Miroslav Krstic and his coauthors (see [24] for a concise introduction).

The backstepping method is now frequently used for various control problems modeling by partial differential equations in one dimension. For example, it has been also used to stabilize the wave equation [23], [35], [32], the parabolic equations in [33], [34], nonlinear parabolic equations [36], and to obtain the null-controllability of the heat equation [12]. The standard backstepping approach relies on the Volterra transform of the second kind. It is worth noting that, in some situations, more general transformations have to be considered as for Korteweg-de Vries equations [4], Kuramoto–Sivashinsky equations [10], Schrödinger’s equation [8], and hyperbolic equations with internal controls [38].

II. STATEMENT OF THE MAIN RESULTS

Define

$$\mathcal{B} := \{B \in \mathbb{R}^{k \times m}; (15) \text{ holds for } 1 \leq i \leq \min\{k, m-1\}\},$$

(13)

and

$$\mathcal{B}_e := \{B \in \mathbb{R}^{k \times m}; (15) \text{ holds for } 1 \leq i \leq k\},$$

(14)

where

the $i \times i$ matrix formed from the last $i$ columns and the last $i$ rows of $B$ is invertible. (15)

We first show that the system (1), (6), and (7) is null-controllable for $B \in \mathcal{B}$ and is exact controllable for $B \in \mathcal{B}_e$ at the time $T_{opt}$ generically. More precisely, we have [13, Theorem 1.1]:

**Theorem 2.1:** Assume that (3) and (4) hold. We have

i) for each $B \in \mathcal{B}$, outside a discrete set of $\gamma$ in $\mathbb{R}$, the control system (1) with $C$ replaced by $\gamma C$, (6), and (7) is null-controllable at the time $T_{opt}$.

ii) for each $\gamma$ outside a discrete set in $\mathbb{R}$, outside a set of zero measure of $B$ in $\mathcal{B}_e$, the control system (1) with $C$ replaced by $\gamma C$, (6), and (7) is null-controllable at the time $T_{opt}$.

**Remark 2.1:** In the case $m = 1$, we can show that there exists a (linear) time independent feedback which yields the null-controllability at the time $T_{opt}$ for (1), (6), and (7) for all $B$, see [13, Theorem 1.1].

**Remark 2.2:** In the case $m = 2$, we also established [13, Theorem 1.1] the following result on the optimality of $T_{opt}$:

If $B \in \mathcal{B}$, $B_{k1} \neq 0$, $\Sigma$ is constant, and $(T_{opt} = \frac{\tau_1 + \tau_2, k \geq 2, \text{ and } T_{opt} = \tau_1 + \tau_2 = \tau_1 + \tau_2 \text{ if } k = 1}$, then there exists a non-zero constant matrix $C$ such that the system is not null-controllable at the time $T_{opt}$.

Concerning the exact controllability, we have

**Theorem 2.2:** Assume that $m \geq k \geq 1$, (3) and (4) hold. We have

i) for each $B \in \mathcal{B}_e$, outside a discrete set of $\gamma$ in $\mathbb{R}$, the control system (1) with $C$ replaced by $\gamma C$, (6), and (7) is exactly controllable at the time $T_{opt}$.

ii) for each $\gamma$ outside a discrete set in $\mathbb{R}$, outside a set of zero measure of $B$ in $\mathcal{B}_e$, the control system (1) with $C$ replaced by $\gamma C$, (6), and (7) is exactly controllable at the time $T_{opt}$.

For the exact controllability and $T > T_{opt}$, the generic assumption was removed in [22] (see also [14, Theorem 4]), where the following theorem is established

**Theorem 2.3:** Let $m \geq k \geq 1$. Assume that $B \in \mathcal{B}_e$ defined in (14). The control system (1), (6), and (7) is exactly controllable at any time $T$ greater than $T_{opt}$.

(In fact [22] gives the optimal time of exact controllability even if $B \notin \mathcal{B}_e$.)

For the null controllability and $T > T_{opt}$, we removed the generic assumption was in [14, Theorems 2], where the following theorem is established
Theorem 2.4: Let $k \geq m \geq 1$. Assume that $B \in \mathcal{B}$ defined in (13). The control system (1), (6), and (7) is null-controllable at any time $T$ greater than $T_{opt}$.

Remark 2.3: Related controllability results can be also found in [20], [37].

Concerning the optimality of $T_{opt}$, we can prove the following result [13, Proposition 1.6]

Proposition 2.1: Assume that $C \equiv 0$ and (15) holds for $1 \leq i \leq \min \{k, m\}$, then, for any $T < T_{opt}$, there exists an initial datum such that $u(T, \cdot) \neq 0$ for every control.

Remark 2.4: The controls used in Theorems 2.1 to 2.4 are of the form

$$w_+(t, 1) = \sum_{r=1}^{R} A_r(t) w(t, x_r) + \int_{0}^{t} M(t, y) w(t, y) dy + h(t),$$

where $R \in \mathbb{N}$, $A_r : [0, T] \rightarrow \mathbb{R}^{m \times n}$, $x_r \in [0, 1]$ $(1 \leq r \leq R)$, $M : [0, T] \times [0, 1] \rightarrow \mathbb{R}^{n \times n}$, and $h \in [L^{\infty}(0, T)]^m$. Moreover, the following conditions hold:

$$x_r < c < 1 \text{ for some constant } c,$$

$$A_r \in [L^{\infty}(0, T)]^{n \times n}, \quad M \in [L^{\infty}((0, T) \times (0, 1))]^{n \times n},$$

for $1 \leq r \leq R$. The well-posedness of (1), (6), and (16) for broad solutions (see [13, Definition 3.1]) was established in [13, Lemma 3.2].

We next discuss homogeneous quasilinear hyperbolic systems. More precisely, we consider the equation, for $(t, x) \in [0, +\infty) \times (0, 1)$,

$$\partial_t w(t, x) = \sum \partial_x w(t, x) \partial_x w(t, x),$$

instead of (1), and the boundary and control conditions (6) and (7). We assume that $\Sigma(x, y)$ for $x \in [0, 1]$ and $y \in \mathbb{R}^n$ is of the form

$$\Sigma(x, y) = \text{diag} \left(-\lambda_1(x, y), \cdots, -\lambda_k(x, y), \lambda_{k+1}(x, y), \cdots, \lambda_{k+m}(x, y)\right),$$

where

$$-\lambda_1(x, y) < \cdots < -\lambda_k(x, y) < 0 < \lambda_{k+1}(x, y) < \cdots < \lambda_{k+m}(x, y).$$

We assume, for $1 \leq i \leq n = k + m$,

$$\lambda_i \text{ is of class } C^2 \text{ with respect to } x \text{ and } y$$

Concerning the quasilinear system (19), (6), and (7), we prove [15, Theorem 1.1]:

**Theorem 2.5:** Assume that $B \in \mathcal{B}$. For any $T > T_{opt}$, there exist $\varepsilon > 0$ and a time-independent feedback control for (19), (6), and (7) such that if the compatibility conditions at $x = 0$ (23) and (24) below hold for $w(0, \cdot)$, and

$$||w(0, \cdot)||_{C^1([0, 1])} < \varepsilon,$$

then $w(T, \cdot) = 0$.

The compatibility conditions considered in Theorem 2.1 are:

$$w_-(0, 0) = B(w_+(0, 0))$$

and

$$\Sigma_-(0, w(0, 0)) \partial_x w_-(0, 0) = B(w_+(0, 0)) \Sigma_+(0, w(0, 0)) \partial_x w_+(0, 0).$$

Here we denote

$$w_+ = (w_1, \cdots, w_k)^T, \quad w_- = (w_{k+1}, \cdots, w_{k+m})^T,$$

$$\Sigma_+(x, y) = \text{diag} \left(-\lambda_1(x, y), \cdots, -\lambda_k(x, y)\right)$$

and

$$\Sigma_-(x, y) = \text{diag} \left(\lambda_{k+1}(x, y), \cdots, \lambda_m(x, y)\right).$$

**Remark 2.5:** In [15], we also consider nonlinear boundary condition at $x = 0$, i.e., instead of (6), we deal with

$$w_-(t, 0) = B(w_+(t, 0)) \text{ for } t \geq 0,$$

for some

$$B \in (C^2(\mathbb{R}^m))^k \text{ with } B(0) = 0,$$

Theorem 2.5 also holds if the condition $B \in \mathcal{B}$ is replaced by $\nabla B(0) \in \mathcal{B}$.

**Remark 2.6:** The feedbacks constructed in [15] use additional $4m$ state-variables (dynamics extensions) to avoid imposing compatibility conditions at $x = 1$.

In our recent work [16], we present Lyapunov’s functions for the feedbacks given in Theorem 2.5 and use estimates for Lyapunov’s functions to rediscover the finite stabilization result.

**III. THE IDEAS OF THE PROOF OF THEOREM 2.1-THEOREM 2.4**

The starting point of our analysis is the backstepping approach. The key idea of the backstepping approach is to make the following change of variables

$$u(t, x) = w(t, x) - \int_{0}^{x} K(x, y) w(t, y) dy,$$

for some kernel $K : \mathcal{T} \rightarrow \mathbb{R}^{n \times n}$ which is chosen in such a way that the system for $u$ is easier to control. Here

$$\mathcal{T} = \{(x, y) \in (0, 1)^2; 0 < y < x\}.$$

To determine/derive the equations for $K$, we first compute $\partial_t u(t, x) - \Sigma(x) \partial_x u(t, x)$. Taking into account (25), we formally have

$$\partial_t u(t, x) = \partial_t w(t, x) - K(x, x) \Sigma(x) w(t, x) + K(x, 0) \Sigma(0) w(t, 0) + \int_{0}^{x} \left[ \partial_t K(x, y) \Sigma(y) w(t, y) - K(x, y) C(y) w(t, y) \right] dy,$$

and

$$\partial_x u(t, x) = \partial_x w(t, x) - \int_{0}^{x} \partial_x K(x, y) w(t, y) dy - K(x, x) w(t, x).$$

1 We assume here that $u$, $w$, and $K$ are smooth enough so that the below computations make sense.
It follows from (1) that
\[ \partial_t u(t,x) - \Sigma(x) \partial_x u(t,x) = \begin{pmatrix} C(x) - K(x,x) \Sigma(x) + \Sigma(x) K(x,x) \\ K(x,0) \Sigma(0) u(t,0) + \int_{\mathbb{R}^2} K(x,y) \Sigma(y) + \Sigma(y) \partial_y K(x,y) \end{pmatrix} w(t,y) + \int_{[L^2(0,1)^n \to L^2(0,1)]} K \text{ from } L^2(0,1)^n \text{ into itself.} \] (27)
Moreover, K depends only on \( \Sigma, S, \) and B; in particular, K is independent of T.

It follows from (1) that \( \partial_t u(t,x) - \Sigma(x) \partial_x u(t,x) = \begin{pmatrix} C(x) - K(x,x) \Sigma(x) + \Sigma(x) K(x,x) \\ K(x,0) \Sigma(0) u(t,0) + \int_{\mathbb{R}^2} K(x,y) \Sigma(y) + \Sigma(y) \partial_y K(x,y) \end{pmatrix} w(t,y) + \int_{[L^2(0,1)^n \to L^2(0,1)]} K \text{ from } L^2(0,1)^n \text{ into itself.} \) Moreover, K depends only on \( \Sigma, S, \) and B; in particular, K is independent of T.

We are able to show that such a K can be chosen in such a way that
\[ K \text{ is the unique solution of the system, for } (t,x) \in (-\infty,0) \times (0,1), \]
\[ \partial v(t,x) = \Sigma(x) \partial_x v(t,x) + \Sigma'(x) v(t,x), \] (39)
with, \( t < 0, \)
\[ v(0,0) = 1, \]
\[ \Sigma_+(0)v_+(t,0) = -B^T \Sigma_-(0)v_-(t,0) + \int_0^1 \Sigma_+(t)x) v_+(t,x) + \Sigma_+(t) \partial_x v(t,x) dx, \] (41)
and
\[ v(t,0) = v(0,0) \text{ in } (0,1). \] (42)

The next key part of the proof of Theorem 2.4 is the following compactness result for (39)-(41) [14, Lemma 4]; the structure of S given in (35) plays a role in the proof.

**Lemma 3.2:** Let \( k \geq m \geq 1, B \in \mathcal{B}, \) and \( T \geq T_{opt}. \) Assume that \( (v_N) \) be a sequence of solutions of (39)-(41) (with \( v_N = 0, \) in \( L^2(0,1)^n \)) such that
\[ \sup_N \|v_N(T,\cdot)\|_{L^2(0,1)} < +\infty, \] (43)
\[ \lim_{N \to +\infty} \|v_N(+\cdot,0)\|_{L^2(-T,0)} = 0. \] (44)

We have, up to a subsequence,
\[ v_N(T,\cdot) \text{ converges in } L^2(0,1), \] (45)
and the limit \( V \in [L^2(0,1)]^n \) satisfies the equation
\[ V = \mathcal{K} V, \] (46)
for some compact operator \( \mathcal{K} \) from \( [L^2(0,1)]^n \) into itself. Moreover, \( \mathcal{K} \) depends only on \( \Sigma, S, \) and \( B; \) in particular, \( \mathcal{K} \) is independent of T.
Proof: [Proof of Theorem 2.4] For $T > T_{opt}$, set

$$Y_T := \left\{ V \in L^2(0,1) : V \text{ is the limit in } L^2(0,1) \text{ of some subsequence of solutions } (v_N(-T,\cdot)) \text{ of (39)-(41) such that (43) and (44) hold} \right\}. \quad (47)$$

It is clear that $Y_T$ is a vectorial space. Moreover, by (46) and the compact property of $\mathcal{X}$, we have

$$\dim Y_T \leq C, \quad (48)$$

for some positive constant $C$ independent of $T$.

We next show that

$$Y_{T_2} \subset Y_{T_1} \text{ for } T_{opt} < T_1 < T_2. \quad (49)$$

Indeed, let $V \in Y_{T_2}$. There exists a sequence of solutions $(v_N)$ of (39)-(41) such that

$$\left\{ \begin{array}{l}
  \{v_N(-T,\cdot) \to V \ \text{in} \ L^2(0,1), \\
  \lim_{N \to +\infty} \|v_N,\cdot,1\|_{L^2(-T_0,0)} = 0.
\end{array} \right. \quad (50)$$

By considering the sequence $v_N(\cdot - \tau,\cdot)$ with $\tau = T_2 - T_1$, we derive that $V \in Y_{T_1}$.

The arguments are then in the spirit of [2] (see also [29]) via an eigenvalue problem in finite dimension using a contradiction argument. By Lemma 3.1, to obtain the null-controllability at the time $T > T_{opt}$, it suffices to prove (38) by contradiction. Assume that there exists a sequence of solutions $(v_N)$ of (39)-(41) such that

$$N \int_{-T}^0 (|v_{N,+}(t,1)|^2) dt \leq \int_{0}^1 |v_N(-T,x)|^2 dx = 1. \quad (51)$$

By (45), up to a subsequence, $v_N(-T,\cdot)$ converges in $L^2(0,1)$ to a limit $V$. It is clear that $\|V\|_{L^2(0,1)} = 1$; in particular, $V \neq 0$. Consequently,

$$Y_T \neq \{0\}. \quad (52)$$

By (48), (49), and (52), there exist $T_{opt} < T_1 < T_2 < T$ such that

$$\dim Y_{T_1} = \dim Y_{T_2} \neq 0. \quad (53)$$

We can prove that, for $V \in Y_{T_1}$,

$$\Sigma \partial_t V + \Sigma' V \text{ is an element of } Y_{T_1}. \quad (54)$$

Recall that $Y_{T_1}$ is real and of finite dimension. Consider its natural extension as a complex vectorial space and still denote its extension by $Y_{T_1}$. Define

$$\mathcal{A} : Y_{T_1} \to Y_{T_1}, \quad V \mapsto \Sigma \partial_t V + \Sigma' V. \quad (55)$$

From the definition of $Y_{T_1}$, it is clear that, for $V \in Y_{T_1}$,

$$V_+(1) = 0 \quad (56)$$

and

$$\Sigma_+(0)V_+(0) = -B^T \Sigma_-(0)V_-(0) + \int_0^1 S_+^T(x)V_-(x) dx. \quad (57)$$

Since $Y_T \neq \{0\}$ and $Y_T$ is of finite dimension, there exists $\lambda \in \mathbb{C}$ and $V \in Y_T \setminus \{0\}$ such that

$$\mathcal{A} V = \lambda V. \quad (58)$$

Set

$$v(t,x) = e^{\lambda t} V(x) \in (-\infty,0) \times \{0,1\}. \quad (59)$$

Using (54) and (55), one can verify that $v(t,x)$ satisfies (39)-(41). Applying the characteristic method, one can deduce that $v(t,) = 0$ in $(0,1)$ for $t < -\tau_k + \cdots - \tau_{k+m}$. It follows that $V = 0$ which contradicts the fact $V \neq 0$. Thus (38) holds and the null-controllability is valid for $T > T_{opt}$. The details can be found in [14].

IV. ON THE PROOF OF THEOREM 2.5

We will also deal with the nonlinear boundary condition at $x = 0$ as mentioned in Remark 2.5. We first consider the case $m > k$. Consider the last equation of (6) and impose the condition $w_k(t,0) = 0$. Using (15) with $i = 1$ and the implicit function theorem, one can then write the last equation of (6) under the form

$$w_{m+k}(t,0) = M_k \left( w_{k+1}(t,0), \ldots, w_{m+k-1}(t,0) \right), \quad (60)$$

for some $C^2$ nonlinear map $M_k$ from $U_k$ into $\mathbb{R}$ for some neighborhood $U_k$ of $0 \in \mathbb{R}^{m-1}$ with $M_k(0) = 0$ provided that $|w_k(t,0)|$ is sufficiently small.

Consider the last two equations of (6) and impose the condition $w_k(t,0) = w_{k-1}(t,0) = 0$. Using (15) with $i = 2$ and the Gaussian elimination approach, one can then write these two equations under the form (60) and

$$w_{m+k-1}(t,0) = M_{k-1} \left( w_{k+1}(t,0), \ldots, w_{m+k-2}(t,0) \right), \quad (61)$$

for some $C^2$ nonlinear map $M_{k-1}$ from $U_{k-1}$ into $\mathbb{R}$ for some neighborhood $U_{k-1}$ of $0 \in \mathbb{R}^{m-2}$ with $M_{k-1}(0) = 0$ provided that $|w_k(t,0)|$ is sufficiently small, etc. Finally, consider the $k$ equations of (6) and impose the condition

$$w_k(t,0) = \cdots = w_1(t,0) = 0. \quad (62)$$

Using (15) with $i = k$ and the Gaussian elimination approach, one can then write these $k$ equations under the form (60), (61), \ldots, and

$$w_{m+1}(t,0) = M_1 \left( w_{k+1}(t,0), \ldots, w_m(t,0) \right), \quad (63)$$

for some $C^2$ nonlinear map $M_1$ from $U_1$ into $\mathbb{R}$ for some neighborhood $U_1$ of $0 \in \mathbb{R}^{m-k}$ with $M_1(0) = 0$ provided that $|w_k(t,0)|$ is sufficiently small. These nonlinear maps $M_1, \ldots, M_k$ will be used in the construction of feedbacks.

Define

$$\frac{d}{dt} x_j(t,s,\xi) = \lambda_j \left( x_j(t,s,\xi), w(t,x_j(t,s,\xi)) \right) \quad (64)$$

with $x_j(s,s,\xi) = \xi$ for $1 \leq j \leq k$, and

$$\frac{d}{dt} x_j(t,s,\xi) = -\lambda_j \left( x_j(t,s,\xi), w(t,x_j(t,s,\xi)) \right), \quad (65)$$

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with \(x_j(s, s, \xi) = \xi\) for \(k + 1 \leq j \leq k + m\). We do not precise at this stage the domain of the definition of \(x_j\). Later, we only consider the flows in the regions where the solution \(w\) is well-defined.

To arrange the compatibility of our controls, we also introduce following auxiliary variables satisfying autonomous dynamics. Set \(\delta = T - T_{opt} > 0\). For \(t \geq 0\), define, for \(k + 1 \leq j \leq k + m\),

\[
\xi_j(0) = w_{0,j}(0), \quad \eta_j(0) = \lambda_j(0, w_0(0))w_{0,j}(0),
\]

and

\[
\eta_j(0) = 1, \quad \eta_j(0) = 0,
\]

\[
\xi_j(t) = \eta_j(t) = 0 \text{ for } t \geq \delta/2.
\]

The feedback is then chosen as follows:

\[
w_{m+k}(t, 1) = \zeta_{m+k}(t)
\]

\[
+ (1 - \eta_{m+k}(t))M_k\left(w_{k+1}(t, x_{k+1}(t, t + t_{m+k}, 0)), \ldots, w_{k+m-1}(t, x_{k+m-1}(t, t + t_{m+k}, 0))\right)
\]

\[
w_{m+k-1}(t, 1) = \zeta_{m+k-1}(t)
\]

\[
+ (1 - \eta_{m+k-1}(t))M_{k-1}\left(w_{k+1}(t, x_{k+1}(t, t + t_{m+k-1}, 0)), \ldots, w_{k+m-2}(t, x_{k+m-2}(t, t + t_{m+k-1}, 0))\right)
\]

\[
\ldots
\]

\[
w_{m+1}(t, 1) = \zeta_{m+1}(t)
\]

\[
+ (1 - \eta_{m+1}(t))M_1\left(w_{k+1}(t, x_{k+1}(t, t + t_{m+1}, 0)), \ldots, w_{m}(t, x_{m}(t, t + t_{m+1}, 0))\right)
\]

and, for \(k + 1 \leq j \leq m\),

\[
w_{j}(t, 1) = \xi_j(t).
\]

This feedback is well-determined by noting that (61) depends only on the current state, (62) depends only on the current state and (61), etc.

For \(m \leq k\), the feedback law is given as follows:

\[
w_{m+k}(t, 1) = \zeta_{m+k}(t)
\]

\[
+ (1 - \eta_{m+k}(t))M_k\left(w_{k+1}(t, x_{k+1}(t, t + t_{m+k}, 0)), \ldots, w_{k+m-1}(t, x_{k+m-1}(t, t + t_{m+k}, 0))\right),
\]

\[
\ldots
\]

\[
w_{k+2}(t, 1) = \zeta_{k+2}(t)
\]

\[
+ (1 - \eta_{k+2}(t))M_2\left(w_{k+1}(t, x_{k+1}(t, t + t_{k+2}, 0))\right),
\]

and

\[
w_{k+1}(t, 1) = \xi_{k+1}(t).
\]

The null-controllability for small initial data can be derived from the properties of \(M_j\) for \(1 \leq j \leq k\). An key technical part of the proof is the well-posedness for (19) and (6) equipped these feedback laws, see [15, Lemma 2.2].

**Conclusion:** This paper is devoted to the null-controllability, exact controllability, and stabilization of hyperbolic systems for the optimal time. The starting point of the analysis in the inhomogeneous case is based on the backstepping approach. The ideas of the analysis are presented.

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