NCO Tracking for Singular Control Problems using Neighboring Extremals ⋆

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Abstract: A powerful approach for dynamic optimization in the presence of uncertainty is to incorporate measurements into the optimization framework so as to track the necessary conditions of optimality (NCO), the so-called NCO-tracking approach. For nonsingular control problems, this can be done by tracking active constraints along boundary arcs, and using neighboring-extremal (NE) control along interior arcs to force the first-order variation of the NCO to zero. In this paper, an extension of NE control to singular control problems is proposed. The idea is to design NE controllers from successive time differentiations of the first-order variation of the NCO. Based on these results, a NCO-tracking controller that is easily tractable from a real-time optimization perspective is proposed, whose application guarantees that the first-order variation of the NCO converges to zero exponentially. The performance of this NCO-tracking controller is illustrated via the case study of a steered car, a 5th-order two-input dynamical system.

1. INTRODUCTION

Optimization in the process industry has received a lot of attention in recent years because, in the face of growing competition, it represents a natural choice for reducing production costs, improving product quality, and meeting safety requirements and environmental regulations. Traditionally, the optimal operating conditions are determined based on a model of the process. However, the resulting process operation can be highly sensitive to uncertainty such as model mismatch and process disturbances. This generally gives rise to suboptimal process operation or, worse, infeasible operation, which of course is not tolerable in most industrial applications.

A natural approach to combat uncertainty and avoid conservatism consists in incorporating measurements in the optimization framework. In particular, the NCO-tracking methodology (Srinivasan and Bonvin, 2007) converts a dynamic optimization problem with both path and terminal constraints into a feedback control problem. In this approach, near-optimal process operation is enforced by tracking appropriate references, namely the *necessary conditions of optimality* (NCO). The idea behind NCO tracking is to take advantage of the structure of an optimal control solution, which is usually made of a succession of arcs. For those arcs along which path constraints are active, part of the optimal inputs are obtained by enforcing the corresponding constraints. The remaining part of the optimal inputs is determined by the intrinsic compromises present in the system. In this latter case, the Pontryagin Maximum Principle (PMP) (Pontryagin et al., 1964) shows that tracking the NCO consists in forcing a sensitivity term to zero. However, this is much more involved than constraints tracking in the sense that the corresponding sensitivity terms depend on the adjoint variables, which are typically unknown and cannot be measured. For nonlinear dynamical systems, a first-order approximation of these sensitivity terms can be obtained upon application of the theory of neighboring extremals (NE) (Bryson and Ho, 1975). In other words, NE control forces the first-order variation of the NCO to zero, and thus offers much promise in the context of NCO tracking.

However, an inherent limitation of standard NE control lies in the fact that the control problem must be nonsingular, otherwise the control law calls for the inversion of a singular matrix. In a previous work, Gros et al. (2004) proposed to design NE controllers for single-input, singular control problems from successive time differentiations of the first-order variation of the NCO. The main contribution of the present paper is to generalize these ideas to the multiple-input case. Most of the complications stem from the fact that an optimal arc for a multiple-input system may have different orders of singularity with respect to the control variables (Robbins, 1967). Based on these new developments, a NCO-tracking controller that is easily implementable and tractable from a real-time optimization perspective is proposed.

The paper is organized as follows. The problem formulation is presented in Section 2. NE control for singular problems is characterized in Section 3, and a multiple-input NCO-tracking controller is devised in Section 4. These new developments are illustrated via the case study of a steered car in Section 5. Finally, Section 6 concludes the paper.

⋆ This material is based upon work supported by the Swiss National Science Foundation under grant 200020-101783.

2. PROBLEM FORMULATION

Consider the following dynamic optimization problem with input bounds:

$$\min J[\mathbf{u}] := \Phi(\mathbf{x}(t_\mathrm{f})) + \int_0^{t_\mathrm{f}} L(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\theta}) \,\mathrm{d}t \qquad (1)$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\theta}); \quad \mathbf{x}(0) = \mathbf{x}_0 \qquad (2)$$

$$\mathbf{u}^L \leq \mathbf{u}(t) \leq \mathbf{u}^U, \qquad (3)$$

where t stands for the time (independent) variable, t_f the fixed final time, $\mathbf{u} : [0, t_\mathrm{f}] \to \mathbb{R}^{n_u}$ the control vector function, $\mathbf{x} : [0, t_\mathrm{f}] \to \mathbb{R}^{n_x}$ the state vector function with initial state $\mathbf{x}_0$, $\boldsymbol{\theta} \in \mathbb{R}^{n_\theta}$ the vector of uncertain time-invariant parameters, $\mathbf{F}$ the system dynamics, J the scalar cost functional to be minimized, Φ the terminal cost function, and L the integral cost function. We shall assume that all functions appearing in (1)–(3) are sufficiently often continuously differentiable with respect to their arguments.

The first- and second-order necessary conditions of optimality for the problem (1)–(3) are given by:

$$H_\mathbf{u} = L_\mathbf{u} + \mathbf{F}_\mathbf{u}^\mathrm{T}\boldsymbol{\lambda} - \boldsymbol{\mu}^L + \boldsymbol{\mu}^U = \mathbf{0} \qquad (4)$$

$$H_\mathbf{uu} \text{ positive semi-definite,}$$

where the Hamiltonian function H is defined as

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}^L, \boldsymbol{\mu}^U) := L(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}) + \mathbf{F}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta})^\mathrm{T}\boldsymbol{\lambda} + {\boldsymbol{\mu}^L}^\mathrm{T}(\mathbf{u}^L - \mathbf{u}) + {\boldsymbol{\mu}^U}^\mathrm{T}(\mathbf{u} - \mathbf{u}^U),$$

$\boldsymbol{\lambda} : [0, t_\mathrm{f}] \to \mathbb{R}^{n_x}$ denotes the adjoint vector function given by

$$\dot{\boldsymbol{\lambda}} = -H_\mathbf{x} = -\mathbf{F}_\mathbf{x}^\mathrm{T}\boldsymbol{\lambda} - L_\mathbf{x}; \quad \boldsymbol{\lambda}(t_\mathrm{f}) = \Phi_\mathbf{x}(\mathbf{x}(t_\mathrm{f})),$$

and $\boldsymbol{\mu}^L, \boldsymbol{\mu}^U : [0, t_\mathrm{f}] \to \mathbb{R}^{n_u}$ are Lagrange multiplier vector functions satisfying

$${\boldsymbol{\mu}^L}^\mathrm{T}(\mathbf{u}^L - \mathbf{u}) = \mathbf{0}; \quad \boldsymbol{\mu}^L \geq \mathbf{0}$$

$${\boldsymbol{\mu}^U}^\mathrm{T}(\mathbf{u} - \mathbf{u}^U) = \mathbf{0}; \quad \boldsymbol{\mu}^U \geq \mathbf{0}.$$

Given the nominal parameter values $\bar{\boldsymbol{\theta}}$, we shall assume that a unique optimal control $\mathbf{u}^*(t)$, $0 \leq t \leq t_\mathrm{f}$, exists in the class of piecewise-continuous vector functions for the optimization problem (1)–(3). We shall call the piece of an optimal trajectory that does not intersect the boundary an *interior arc*; if at least one constraint is active, we call that piece of trajectory a *boundary arc*. Observe that $\boldsymbol{\mu}^L(t) = \boldsymbol{\mu}^U(t) = \mathbf{0}$ along an interior arc, while there is some $i \in \{1, \ldots, n_u\}$ such that $\mu_i^L(t) \neq 0$ or $\mu_i^U(t) \neq 0$ along a boundary arc.

In optimal control theory, singular control problems are those for which a straightforward application of the foregoing NCO fails to provide adequate tests for singling out optimal control values (Bell and Jacobson, 1975). In other words, the matrix $H_\mathbf{uu}$ is singular. To determine optimal control values along singular arcs, one usually exploits the identical vanishing of $H_\mathbf{u}$ and its successive time derivatives $\dot{H}_\mathbf{u}, \ddot{H}_\mathbf{u}, \ldots$

3. CHARACTERIZATION OF NE CONTROL

Any slight change $\eta\delta\mathbf{x}_0$ in the initial state or $\eta\delta\boldsymbol{\theta}$ in the model parameters modifies the optimal control trajectory $\mathbf{u}^*(t)$, $0 \leq t \leq t_\mathrm{f}$, and requires that it be recalculated. Clearly, calculating a perturbed optimal control is a time-consuming task, hardly compatible with the objective of real-time optimization. A more tractable way of obtaining a perturbed optimal control trajectory is to consider the first-order approximation

$$\mathbf{u}(t; \eta) = \mathbf{u}^*(t) + \eta\delta\mathbf{u}(t) + o(\eta),$$

and then use the theory of neighboring extremals for calculating the correction $\delta\mathbf{u}$ in such a way that the first-order variation of the NCO be equal to zero upon application of the control $\mathbf{u}^*(t) + \eta\delta\mathbf{u}(t)$.

Let $(\mathbf{u}^*(t), \mathbf{x}^*(t), \boldsymbol{\lambda}^*(t))$, $0 \leq t \leq t_\mathrm{f}$, be an optimal triple for the optimal control problem (1)–(3) corresponding to the nominal parameter values $\bar{\boldsymbol{\theta}}$. Along each arc composing $\mathbf{u}^*$, a control variable $u_i^*(t)$ may either:

- belong to the interior of the control region $u_i^L < u_i^*(t) < u_i^U$, in which case a neighboring-extremal solution is such that $\delta\mu_i^L(t) = \delta\mu_i^U(t) = 0$, and $\delta u_i(t)$ is obtained from the first variation of (4) as
$$\delta H_{u_i} = H^*_{u_i\mathbf{x}}\delta\mathbf{x} + \mathbf{F}^{*\mathrm{T}}_{u_i}\delta\boldsymbol{\lambda} + H^*_{\mathbf{u}u_i}\delta u_i + H^*_{u_i\boldsymbol{\theta}}\delta\boldsymbol{\theta} = 0;$$
- or be at one of its boundaries u_i^L or u_i^U, in which case a NE control is simply given by $\delta u_i(t) = 0$.

These n_u conditions can be written collectively in the form

$$\delta\boldsymbol{\mathcal{L}} := \mathbf{A}_0\delta\boldsymbol{\lambda} + \mathbf{B}_0\delta\mathbf{x} + \mathbf{C}_0\delta\mathbf{u} + \mathbf{D}_0\delta\boldsymbol{\theta} = \mathbf{0}, \qquad (5)$$

where $\mathbf{A}_0(t), \mathbf{B}_0(t) \in \mathbb{R}^{n_u \times n_x}$, $\mathbf{C}_0(t) \in \mathbb{R}^{n_u \times n_u}$, $\mathbf{D}_0(t) \in \mathbb{R}^{n_u \times n_\theta}$, and

$$\delta\dot{\mathbf{x}} = \mathbf{F}^*_\mathbf{x}\delta\mathbf{x} + \mathbf{F}^*_\mathbf{u}\delta\mathbf{u} + \mathbf{F}^*_{\boldsymbol{\theta}}\delta\boldsymbol{\theta}$$

$$\delta\dot{\boldsymbol{\lambda}} = -\mathbf{F}^{*\mathrm{T}}_\mathbf{x}\delta\boldsymbol{\lambda} - H^*_\mathbf{xx}\delta\mathbf{x} - H^*_\mathbf{xu}\delta\mathbf{u} - H^*_{\mathbf{x}\boldsymbol{\theta}}\delta\boldsymbol{\theta}$$

$$\delta\mathbf{x}(0) = \delta\mathbf{x}_0, \qquad \delta\boldsymbol{\lambda}(t_\mathrm{f}) = \Phi^*_\mathbf{xx}\delta\mathbf{x}(t_\mathrm{f}).$$

If $\mathbf{C}_0$ has full rank, a NE control law is readily obtained from (5) as

$$\delta\mathbf{u} = -\mathbf{C}_0^{-1}\left[\mathbf{A}_0\delta\boldsymbol{\lambda} + \mathbf{B}_0\delta\mathbf{x} + \mathbf{D}_0\delta\boldsymbol{\theta}\right],$$

which corresponds to the standard NE control law in the case where no input constraint is active (Bryson and Ho, 1975).

On the other hand, if $\mathbf{C}_0$ is singular and of constant rank $(n_u - r_0)$, singular value decomposition (SVD) of $\mathbf{C}_0$ gives

$$\begin{pmatrix} \mathbf{U}_0^\mathrm{ns} & \mathbf{U}_0^\mathrm{s} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Sigma}_0 & \\ & \mathbf{0}_{r_0 \times r_0} \end{pmatrix} \begin{pmatrix} {\mathbf{V}_0^\mathrm{ns}}^\mathrm{T} \\ {\mathbf{V}_0^\mathrm{s}}^\mathrm{T} \end{pmatrix} := \mathbf{C}_0,$$

where $\boldsymbol{\Sigma}_0$ is a diagonal, positive-definite matrix, and $\mathbf{U}_0 := \begin{pmatrix} \mathbf{U}_0^\mathrm{ns} & \mathbf{U}_0^\mathrm{s} \end{pmatrix}$, $\mathbf{V}_0 := \begin{pmatrix} \mathbf{V}_0^\mathrm{ns} & \mathbf{V}_0^\mathrm{s} \end{pmatrix}$ are orthogonal matrices. The input variations $\delta\mathbf{u}(t)$ can then be partitioned into nonsingular $\delta\mathbf{u}_0^\mathrm{ns}(t)$ and singular $\delta\mathbf{u}_0^\mathrm{s}(t)$ subparts of dimension $(n_u - r_0)$ and r_0, respectively,

$$\delta\mathbf{u}_0^\mathrm{ns}(t) = {\mathbf{V}_0^\mathrm{ns}}^\mathrm{T}(t)\ \delta\mathbf{u}(t), \qquad \delta\mathbf{u}_0^\mathrm{s}(t) = {\mathbf{V}_0^\mathrm{s}}^\mathrm{T}(t)\ \delta\mathbf{u}(t).$$

Using this partition, the conditions (5) can be split into

$$\mathbf{0} = {\mathbf{U}_0^\mathrm{ns}}^\mathrm{T}\delta\boldsymbol{\mathcal{L}} = {\mathbf{U}_0^\mathrm{ns}}^\mathrm{T}\left(\mathbf{A}_0\delta\boldsymbol{\lambda} + \mathbf{B}_0\delta\mathbf{x} + \mathbf{D}_0\delta\boldsymbol{\theta}\right) + \boldsymbol{\Sigma}_0\delta\mathbf{u}_0^\mathrm{ns}$$

$$\mathbf{0} = {\mathbf{U}_0^\mathrm{s}}^\mathrm{T}\delta\boldsymbol{\mathcal{L}} = {\mathbf{U}_0^\mathrm{s}}^\mathrm{T}\left(\mathbf{A}_0\delta\boldsymbol{\lambda} + \mathbf{B}_0\delta\mathbf{x} + \mathbf{D}_0\delta\boldsymbol{\theta}\right).$$

Matrix $\boldsymbol{\Sigma}_0$ being invertible, the former $(n_u - r_0)$ conditions provide an explicit expression for the nonsingular control variations $\delta\mathbf{u}_0^\mathrm{ns}$ as

$$\delta\mathbf{u}_0^\mathrm{ns} = -\boldsymbol{\Sigma}_0^{-1}{\mathbf{U}_0^\mathrm{ns}}^\mathrm{T}\left(\mathbf{A}_0\delta\boldsymbol{\lambda} + \mathbf{B}_0\delta\mathbf{x} + \mathbf{D}_0\delta\boldsymbol{\theta}\right).$$

The remaining r_0 singular control variations $\delta\mathbf{u}_0^\mathrm{s}$ are determined from the latter r_0 conditions. Introducing the new variables $\delta\boldsymbol{\ell}_0 := \mathbf{U}_0^{\mathrm{s}}{}^{\mathrm{T}}\delta\boldsymbol{\mathcal{L}} \in \mathbb{R}^{r_0}$, and differentiating $\delta\boldsymbol{\ell}_0$ twice with respect to time leads to $2r_0$ additional conditions of the form

$$\begin{aligned}\mathbf{0} &= \delta\dot{\boldsymbol{\ell}}_0 = \mathbf{A}_1\delta\boldsymbol{\lambda} + \mathbf{B}_1\delta\mathbf{x} + \mathbf{D}_1\delta\boldsymbol{\theta}\\ \mathbf{0} &= \delta\ddot{\boldsymbol{\ell}}_0 = \mathbf{A}_2\delta\boldsymbol{\lambda} + \mathbf{B}_2\delta\mathbf{x} + \mathbf{C}_2\delta\mathbf{u}_0^{\mathrm{s}} + \mathbf{D}_2\delta\boldsymbol{\theta}, \end{aligned} \tag{6}$$

with $\mathbf{A}_1, \mathbf{B}_1, \mathbf{A}_2, \mathbf{B}_2 \in \mathbb{R}^{r_0\times n_x}$, $\mathbf{D}_1, \mathbf{D}_2 \in \mathbb{R}^{r_0\times n_\theta}$, and $\mathbf{C}_2 \in \mathbb{R}^{r_0\times r_0}$.

At this point, we have either one of two cases:

- If $\mathbf{C}_2$ has full rank, the singular NE control $\delta\mathbf{u}_0^{\mathrm{s}}(t)$ is obtained from (6) as
$$\delta\mathbf{u}_0^{\mathrm{s}} = -\mathbf{C}_2^{-1}\left(\mathbf{A}_2\delta\boldsymbol{\lambda} + \mathbf{B}_2\delta\mathbf{x} + \mathbf{D}_2\delta\boldsymbol{\theta}\right).$$
The NE control law is then obtained by piecing singular and nonsingular control variations together as
$$\begin{aligned}\delta\mathbf{u} = &-\mathbf{V}_0^{\mathrm{ns}}\boldsymbol{\Sigma}_0^{-1}\mathbf{U}_0^{\mathrm{ns}\,\mathrm{T}}\left(\mathbf{A}_0\delta\boldsymbol{\lambda} + \mathbf{B}_0\delta\mathbf{x} + \mathbf{D}_0\delta\boldsymbol{\theta}\right)\\ &-\mathbf{V}_0^{\mathrm{s}}\mathbf{C}_2^{-1}\left(\mathbf{A}_2\delta\boldsymbol{\lambda} + \mathbf{B}_2\delta\mathbf{x} + \mathbf{D}_2\delta\boldsymbol{\theta}\right);\end{aligned}$$
- If $\mathbf{C}_2$ is singular and of constant rank $(r_0 - r_2)$, one uses SVD of $\mathbf{C}_2$ to further partition the singular control variations as $\delta\mathbf{u}_2^{\mathrm{ns}} := \mathbf{V}_2^{\mathrm{ns}\,\mathrm{T}}\delta\mathbf{u}_0^{\mathrm{s}} \in \mathbb{R}^{r_0-r_2}$ and $\delta\mathbf{u}_2^{\mathrm{s}} := \mathbf{V}_2^{\mathrm{s}}{}^{\mathrm{T}}\delta\mathbf{u}_0^{\mathrm{s}} \in \mathbb{R}^{r_2}$. Then, introducing the new variables $\delta\boldsymbol{\ell}_2 := \mathbf{U}_2^{\mathrm{s}}{}^{\mathrm{T}}\delta\ddot{\boldsymbol{\ell}}_0 \in \mathbb{R}^{r_2}$, one can proceed as before by differentiating $\delta\boldsymbol{\ell}_2$ twice, and the procedure continues in an obvious manner.

By continuing the recursion outlined previously, that is

$$\delta\boldsymbol{\ell}_{2k} := \mathbf{U}_{2k}^{\mathrm{s}}{}^{\mathrm{T}}\delta\ddot{\boldsymbol{\ell}}_{2k-2} \in \mathbb{R}^{r_{2k}}; \qquad \delta\boldsymbol{\ell}_0 := \mathbf{U}_0^{\mathrm{s}}{}^{\mathrm{T}}\delta\boldsymbol{\mathcal{L}}, \tag{7}$$

one obtains the following set of $3r_{2k-2}$ conditions that must hold along a singular arc:

$$\begin{aligned}\mathbf{0} &= \delta\boldsymbol{\ell}_{2k-2} = \mathbf{U}_{2k-2}^{\mathrm{s}}{}^{\mathrm{T}}\left(\mathbf{A}_{2k-2}\delta\boldsymbol{\lambda} + \mathbf{B}_{2k-2}\delta\mathbf{x} + \mathbf{D}_{2k-2}\delta\boldsymbol{\theta}\right)\\ \mathbf{0} &= \delta\dot{\boldsymbol{\ell}}_{2k-2} = \mathbf{A}_{2k-1}\delta\boldsymbol{\lambda} + \mathbf{B}_{2k-1}\delta\mathbf{x} + \mathbf{D}_{2k-1}\delta\boldsymbol{\theta}\\ \mathbf{0} &= \delta\ddot{\boldsymbol{\ell}}_{2k-2} = \mathbf{A}_{2k}\delta\boldsymbol{\lambda} + \mathbf{B}_{2k}\delta\mathbf{x} + \mathbf{C}_{2k}\delta\mathbf{u}_{2k-2}^{\mathrm{s}} + \mathbf{D}_{2k}\delta\boldsymbol{\theta},\end{aligned}$$

where $\mathbf{A}_{2k-1}, \mathbf{B}_{2k-1}, \mathbf{A}_{2k}, \mathbf{B}_{2k} \in \mathbb{R}^{r_{2k-2}\times n_x}$, $\mathbf{D}_{2k-1}, \mathbf{D}_{2k} \in \mathbb{R}^{r_{2k-2}\times n_\theta}$, $\mathbf{C}_{2k} \in \mathbb{R}^{r_{2k-2}\times r_{2k-2}}$ are defined recursively as

$$\begin{aligned}\mathbf{A}_{2k-1} = &\ \dot{\mathbf{U}}_{2k-2}^{\mathrm{sT}}\mathbf{A}_{2k-2}\\ &+ \mathbf{U}_{2k-2}^{\mathrm{sT}}\left(\dot{\mathbf{A}}_{2k-2} - \mathbf{A}_{2k-2}\mathbf{F}_{\mathbf{x}}^{*\mathrm{T}}\right)\end{aligned} \tag{8}$$

$$\begin{aligned}\mathbf{B}_{2k-1} = &\ \dot{\mathbf{U}}_{2k-2}^{\mathrm{sT}}\mathbf{B}_{2k-2}\\ &+ \mathbf{U}_{2k-2}^{\mathrm{sT}}\left(\dot{\mathbf{B}}_{2k-2} + \mathbf{B}_{2k-2}\mathbf{F}_{\mathbf{x}}^{*} - \mathbf{A}_{2k-2}H_{\mathbf{xx}}^{*}\right)\end{aligned} \tag{9}$$

$$\begin{aligned}\mathbf{D}_{2k-1} = &\ \dot{\mathbf{U}}_{2k-2}^{\mathrm{sT}}\mathbf{D}_{2k-2}\\ &+ \mathbf{U}_{2k-2}^{\mathrm{sT}}\left(\dot{\mathbf{D}}_{2k-2} + \mathbf{B}_{2k-2}\mathbf{F}_{\boldsymbol{\theta}}^{*} - \mathbf{A}_{2k-2}H_{\mathbf{x}\boldsymbol{\theta}}^{*}\right)\end{aligned} \tag{10}$$

$$\begin{aligned}\mathbf{A}_{2k} = &\ \dot{\mathbf{A}}_{2k-1} - \mathbf{A}_{2k-1}\mathbf{F}_{\mathbf{x}}^{*\mathrm{T}}\\ &- \left(\mathbf{B}_{2k-1}\mathbf{F}_{\mathbf{u}}^{*} - \mathbf{A}_{2k-1}H_{\mathbf{xu}}^{*}\right)\\ &\times\left[\sum_{i=0}^{k-1}\left(\prod_{j=0}^{i-1}\mathbf{V}_{2j}^{\mathrm{s}}\right)\left(\mathbf{V}_{2i}^{\mathrm{ns}}\boldsymbol{\Sigma}_{2i}^{-1}\mathbf{U}_{2i}^{\mathrm{nsT}}\right)\mathbf{A}_{2i}\right]\end{aligned} \tag{11}$$

$$\begin{aligned}\mathbf{B}_{2k} = &\ \dot{\mathbf{B}}_{2k-1} + \mathbf{B}_{2k-1}\mathbf{F}_{\mathbf{x}}^{*} - \mathbf{A}_{2k-1}H_{\mathbf{xx}}^{*}\\ &- \left(\mathbf{B}_{2k-1}\mathbf{F}_{\mathbf{u}}^{*} - \mathbf{A}_{2k-1}H_{\mathbf{xu}}^{*}\right)\\ &\times\left[\sum_{i=0}^{k-1}\left(\prod_{j=0}^{i-1}\mathbf{V}_{2j}^{\mathrm{s}}\right)\left(\mathbf{V}_{2i}^{\mathrm{ns}}\boldsymbol{\Sigma}_{2i}^{-1}\mathbf{U}_{2i}^{\mathrm{nsT}}\right)\mathbf{B}_{2i}\right]\end{aligned} \tag{12}$$

$$\mathbf{C}_{2k} = \left(\mathbf{B}_{2k-1}\mathbf{F}_{\mathbf{u}}^{*} - \mathbf{A}_{2k-1}H_{\mathbf{xu}}^{*}\right)\left(\prod_{j=0}^{k-1}\mathbf{V}_{2j}^{\mathrm{s}}\right) \tag{13}$$

$$\begin{aligned}\mathbf{D}_{2k} = &\ \dot{\mathbf{D}}_{2k-1} + \mathbf{B}_{2k-1}\mathbf{F}_{\boldsymbol{\theta}}^{*} - \mathbf{A}_{2k-1}H_{\mathbf{x}\boldsymbol{\theta}}^{*}\\ &- \left(\mathbf{B}_{2k-1}\mathbf{F}_{\mathbf{u}}^{*} - \mathbf{A}_{2k-1}H_{\mathbf{xu}}^{*}\right)\\ &\times\left[\sum_{i=0}^{k-1}\left(\prod_{j=0}^{i-1}\mathbf{V}_{2j}^{\mathrm{s}}\right)\left(\mathbf{V}_{2i}^{\mathrm{ns}}\boldsymbol{\Sigma}_{2i}^{-1}\mathbf{U}_{2i}^{\mathrm{nsT}}\right)\mathbf{D}_{2i}\right],\end{aligned} \tag{14}$$

and the matrices $\mathbf{U}_{2k}^{\mathrm{ns}}$, $\mathbf{U}_{2k}^{\mathrm{s}}$, $\mathbf{V}_{2k}^{\mathrm{ns}}$ and $\mathbf{V}_{2k}^{\mathrm{s}}$ are obtained from SVD of $\mathbf{C}_{2k}$ as

$$\left(\mathbf{U}_{2k}^{\mathrm{ns}}\ \mathbf{U}_{2k}^{\mathrm{s}}\right)\begin{pmatrix}\boldsymbol{\Sigma}_{2k} & \\ & \mathbf{0}_{r_{2k}\times r_{2k}}\end{pmatrix}\begin{pmatrix}\mathbf{V}_{2k}^{\mathrm{ns}\,\mathrm{T}}\\ \mathbf{V}_{2k}^{\mathrm{s}}{}^{\mathrm{T}}\end{pmatrix} := \mathbf{C}_{2k}. \tag{15}$$

It is assumed throughout that the rank $(r_{2k-2} - r_{2k})$ of the matrix $\mathbf{C}_{2k}$ is constant along the singular arc for each $k = 0, 1, \ldots$

The procedure is stopped after p iterations, where p stands for the smallest value of k such that $\mathbf{C}_{2p}$ is nonsingular. A finite value is assumed for p in this work, i.e. the control problem is not degenerate. Then, the control variation $\delta\mathbf{u}(t)$ is obtained by piecing nonsingular and singular control variations together,

$$\delta\mathbf{u} = \sum_{k=0}^{p-1}\left(\prod_{i=0}^{k-1}\mathbf{V}_{2i}^{\mathrm{s}}\right)\mathbf{V}_{2k}^{\mathrm{ns}}\delta\mathbf{u}_{2k}^{\mathrm{ns}} + \left(\prod_{i=0}^{p-1}\mathbf{V}_{2i}^{\mathrm{s}}\right)\delta\mathbf{u}_{2p-2}^{\mathrm{s}}, \tag{16}$$

where the control variations $\delta\mathbf{u}_0^{\mathrm{ns}}(t), \ldots, \delta\mathbf{u}_{2p-2}^{\mathrm{ns}}(t)$, and $\delta\mathbf{u}_{2p-2}^{\mathrm{s}}(t)$ are obtained from the conditions $\delta\ddot{\boldsymbol{\ell}}_{2k-2} = \mathbf{0}$ as

$$\begin{aligned}\delta\mathbf{u}_{2k}^{\mathrm{ns}} &= -\boldsymbol{\Sigma}_{2k}^{-1}\mathbf{U}_{2k}^{\mathrm{ns}\,\mathrm{T}}\left(\mathbf{A}_{2k}\delta\boldsymbol{\lambda} + \mathbf{B}_{2k}\delta\mathbf{x} + \mathbf{D}_{2k}\delta\boldsymbol{\theta}\right)\\ \delta\mathbf{u}_{2p-2}^{\mathrm{s}} &= -\mathbf{C}_{2p}^{-1}\left(\mathbf{A}_{2p}\delta\boldsymbol{\lambda} + \mathbf{B}_{2p}\delta\mathbf{x} + \mathbf{D}_{2p}\delta\boldsymbol{\theta}\right).\end{aligned}$$

4. DESIGN OF NCO-TRACKING CONTROLLERS

In general, the junction times between the various arcs constituting the optimal solution $\mathbf{u}^*(t)$, $0 \le t \le t_{\mathrm{f}}$, vary when initial conditions or model parameters are perturbed. For many control problems, however, these variations remain small and can be ignored without much effect on system performance. This is the case, e.g., when the nominal solution is dominated by a small number of (possibly singular) arcs. We shall see in this section how simple NCO-tracking controllers can be devised for such problems by fixing the junction times to their nominal values. The resulting controllers are advantageous from a practical viewpoint because the state feedback law comes as a closed-form expression, i.e. it does not necessitate time-consuming on-line computations (such as the on-line solution of a TPBVP).

By construction, applying the feedback law (16) guarantees that the conditions $\mathbf{U}_0^{\mathrm{ns}\,\mathrm{T}}\delta\boldsymbol{\mathcal{L}} = \mathbf{0}$, $\mathbf{U}_{2k}^{\mathrm{ns}\,\mathrm{T}}\delta\ddot{\boldsymbol{\ell}}_{2k-2} = \mathbf{0}$, $k = 1, \ldots, p-1$, and $\delta\ddot{\boldsymbol{\ell}}_{2p-2} = \mathbf{0}$ are satisfied along a singular arc. Yet, the remaining conditions $\mathbf{U}_0^{\mathrm{s}}{}^{\mathrm{T}}\delta\boldsymbol{\mathcal{L}} = \mathbf{0}$, $\mathbf{U}_{2k}^{\mathrm{s}}{}^{\mathrm{T}}\delta\ddot{\boldsymbol{\ell}}_{2k-2} = \mathbf{0}$ and $\delta\dot{\boldsymbol{\ell}}_{2k-2} = \mathbf{0}$, $k = 1, \ldots, p$, are not enforced by (16) and may be violated when the junction times are fixed; in particular, this leads to a violation of the first-order variation of the NCO, $\delta\boldsymbol{\mathcal{L}} = \mathbf{0}$.

A way of enforcing this latter condition is to modify the recursion (7) as

$$\delta \boldsymbol{\ell}_{2k} := {\mathbf{U}^{\mathrm{s}}_{2k}}^{\mathrm{T}} T_{2k-2} \delta \boldsymbol{\ell}_{2k-2}; \qquad \delta \boldsymbol{\ell}_0 := {\mathbf{U}^{\mathrm{s}}_0}^{\mathrm{T}} \delta \boldsymbol{\mathcal{L}},$$

where the operator T_{2k-2} is given by

$$T_{2k-2} := \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \gamma^1_{2k-2} \frac{\mathrm{d}}{\mathrm{d}t} + \gamma^0_{2k-2}, \tag{17}$$

and $\gamma^0_{2k-2}, \gamma^1_{2k-2} \in \mathbb{R}$ are constant gain coefficients. By finite induction on $k = 1, \ldots, p-1$, it can be shown that this modification leads to the following expressions:

$$\begin{aligned}
\delta \boldsymbol{\ell}_{2k} &= {\mathbf{U}^{\mathrm{s}}_{2k}}^{\mathrm{T}} \left(\bar{\mathbf{A}}_{2k} \delta \boldsymbol{\lambda} + \bar{\mathbf{B}}_{2k} \delta \mathbf{x} + \bar{\mathbf{D}}_{2k} \delta \boldsymbol{\theta} \right) \\
\delta \dot{\boldsymbol{\ell}}_{2k} &= \mathbf{A}_{2k+1} \delta \boldsymbol{\lambda} + \mathbf{B}_{2k+1} \delta \mathbf{x} + \mathbf{D}_{2k+1} \delta \boldsymbol{\theta} \\
\delta \ddot{\boldsymbol{\ell}}_{2k} &= \mathbf{A}_{2k+2} \delta \boldsymbol{\lambda} + \mathbf{B}_{2k+2} \delta \mathbf{x} + \mathbf{C}_{2k+2} \delta \mathbf{u}^{\mathrm{s}}_{2k} + \mathbf{D}_{2k+2} \delta \boldsymbol{\theta},
\end{aligned}$$

where:

$$\begin{aligned}
\bar{\mathbf{A}}_{2k} &= \mathbf{A}_{2k} + \gamma^1_{2k-2} \mathbf{A}_{2k-1} + \gamma^0_{2k-2} {\mathbf{U}^{\mathrm{s}}_{2k-2}}^{\mathrm{T}} \mathbf{A}_{2k-2} \\
\bar{\mathbf{B}}_{2k} &= \mathbf{B}_{2k} + \gamma^1_{2k-2} \mathbf{B}_{2k-1} + \gamma^0_{2k-2} {\mathbf{U}^{\mathrm{s}}_{2k-2}}^{\mathrm{T}} \mathbf{B}_{2k-2} \\
\bar{\mathbf{D}}_{2k} &= \mathbf{D}_{2k} + \gamma^1_{2k-2} \mathbf{D}_{2k-1} + \gamma^0_{2k-2} {\mathbf{U}^{\mathrm{s}}_{2k-2}}^{\mathrm{T}} \mathbf{D}_{2k-2} \\
\bar{\mathbf{A}}_0 &= \mathbf{A}_0, \qquad \bar{\mathbf{B}}_0 = \mathbf{B}_0, \qquad \bar{\mathbf{D}}_0 = \mathbf{D}_0.
\end{aligned}$$

The matrices $\mathbf{A}_{2k+1}, \mathbf{B}_{2k+1}, \mathbf{D}_{2k+1}$ are calculated as in (8)–(10); $\mathbf{A}_{2k}, \mathbf{B}_{2k}, \mathbf{C}_{2k}, \mathbf{D}_{2k}$ are calculated as in (11)–(14) except that $\mathbf{A}_{2i}$, $\mathbf{B}_{2i}$, and $\mathbf{D}_{2i}$ are now replaced by $\bar{\mathbf{A}}_{2i}$, $\bar{\mathbf{B}}_{2i}$, and $\bar{\mathbf{D}}_{2i}$; the matrices $\mathbf{U}^{\mathrm{ns}}_{2k}$, $\mathbf{U}^{\mathrm{s}}_{2k}$, $\mathbf{V}^{\mathrm{ns}}_{2k}$ and $\mathbf{V}^{\mathrm{s}}_{2k}$ are obtained from SVD of $\mathbf{C}_{2k}$ as in (15).

Observe that the expression of $\delta \ddot{\boldsymbol{\ell}}_{2k-2}$ being the same as in Section 3, the expression of the new feedback is identical to (16), with the modified matrices $\bar{\mathbf{A}}_{2k}, \bar{\mathbf{B}}_{2k}, \bar{\mathbf{D}}_{2k}$. This feedback law can be rewritten in more compact form

$$\delta \mathbf{u}(t) = -\boldsymbol{\mathcal{C}}(t) \left[\boldsymbol{\mathcal{A}}(t) \delta \boldsymbol{\lambda}(t) + \boldsymbol{\mathcal{B}}(t) \delta \mathbf{x}(t) + \boldsymbol{\mathcal{D}}(t) \delta \boldsymbol{\theta} \right], \tag{18}$$

with the matrices $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}} \in \mathbb{R}^{(r_0 + \cdots + r_{2p}) \times n_x}$, $\boldsymbol{\mathcal{D}} \in \mathbb{R}^{(r_0 + \cdots + r_{2p}) \times n_p}$, and $\boldsymbol{\mathcal{C}} \in \mathbb{R}^{n_u \times (r_0 + \cdots + r_{2p})}$ given by:

$$\boldsymbol{\mathcal{A}} := \begin{pmatrix} \bar{\mathbf{A}}_0 \\ \bar{\mathbf{A}}_2 \\ \vdots \\ \bar{\mathbf{A}}_{2p} \end{pmatrix}, \qquad \boldsymbol{\mathcal{B}} := \begin{pmatrix} \bar{\mathbf{B}}_0 \\ \bar{\mathbf{B}}_2 \\ \vdots \\ \bar{\mathbf{B}}_{2p} \end{pmatrix}, \qquad \boldsymbol{\mathcal{D}} := \begin{pmatrix} \bar{\mathbf{D}}_0 \\ \bar{\mathbf{D}}_2 \\ \vdots \\ \bar{\mathbf{D}}_{2p} \end{pmatrix},$$

$$\boldsymbol{\mathcal{C}} := \left(\cdots \prod_{i=0}^{k-1} \mathbf{V}^{\mathrm{s}}_{2i} \left(\mathbf{V}^{\mathrm{ns}}_{2k} \boldsymbol{\Sigma}^{-1}_{2k} {\mathbf{U}^{\mathrm{ns}}_{2k}}^{\mathrm{T}} \right) \cdots \prod_{i=0}^{p-1} \mathbf{V}^{\mathrm{s}}_{2i} \, \mathbf{C}^{-1}_{2p} \right).$$

The following theorem shows that, under mild assumptions, the modified NE feedback law (18) drives the first-order variation of the NCO to zero, in response to variations in both the initial condition $\delta \mathbf{x}_0$ and the model parameters $\delta \boldsymbol{\theta}$.

Theorem 1. Let $\Gamma_{2k} \in \mathbb{R}^{2 \times 2}$ be defined as

$$\Gamma_{2k} := \begin{pmatrix} 0 & 1 \\ -\gamma^0_{2k} & -\gamma^1_{2k} \end{pmatrix}.$$

If Γ_{2k} is Hurwitz for each $k = 0, \ldots, p-1$, then the first variation $\delta \boldsymbol{\mathcal{L}}$ of the NCO converges to zero exponentially upon application of the feedback law (18).

Proof. See Gros (2007) for a proof.

The gain coefficients $\gamma^0_{2k}, \gamma^1_{2k}$, $0 \leq k < p$, determine the rate of convergence of $\delta \boldsymbol{\mathcal{L}}$ and must be selected carefully. While too small values may not allow to reject the perturbations sufficiently rapidly relative to the time horizon $[0, t_{\mathrm{f}}]$, large values may lead to excessive corrections that invalidate the linear approximation and make the feedback highly sensitive to measurement noise. As a possible extension, one may consider gain matrices $\boldsymbol{\Gamma}^0_{2k}, \boldsymbol{\Gamma}^1_{2k} \in \mathbb{R}^{r_{2k} \times r_{2k}}$ instead of scalar gain coefficients. It can be shown that the result in Theorem 1 still holds, provided that the $(2r_{2k} \times 2r_{2k})$ matrix $\begin{pmatrix} \mathbf{0}_{r_{2k} \times r_{2k}} & \mathbf{I}_{r_{2k} \times r_{2k}} \\ -\boldsymbol{\Gamma}^0_{2k} & -\boldsymbol{\Gamma}^1_{2k} \end{pmatrix}$ is Hurwitz, for each $k = 0, \ldots, p-1$.

Finally, an explicit feedback law is obtained from (18) based on the backward sweep method (Bryson and Ho, 1975):

$$\begin{aligned}
\delta \mathbf{u}(t) &= -\mathbf{K}_{\mathbf{x}}(t) \delta \mathbf{x}(t) - \mathbf{K}_{\boldsymbol{\theta}}(t) \delta \boldsymbol{\theta} && (19) \\
\mathbf{K}_{\mathbf{x}}(t) &= \boldsymbol{\mathcal{C}}(t) \left[\boldsymbol{\mathcal{A}}(t) \mathbf{S}_{\mathbf{x}}(t) + \boldsymbol{\mathcal{B}}(t) \right] && (20) \\
\mathbf{K}_{\boldsymbol{\theta}}(t) &= \boldsymbol{\mathcal{C}}(t) \left[\boldsymbol{\mathcal{A}}(t) \mathbf{S}_{\boldsymbol{\theta}}(t) + \boldsymbol{\mathcal{D}}(t) \right] && (21) \\
\dot{\mathbf{S}}_{\mathbf{x}}(t) &= -H^*_{\mathbf{xx}} - \mathbf{S}_{\mathbf{x}}(t) \mathbf{F}^*_{\mathbf{x}} - {\mathbf{F}^*_{\mathbf{x}}}^{\mathrm{T}} \mathbf{S}_{\mathbf{x}}(t) && (22) \\
&\quad + \left[H^*_{\mathbf{xu}} + \mathbf{S}_{\mathbf{x}}(t) \mathbf{F}^*_{\mathbf{u}} \right] \mathbf{K}_{\mathbf{x}}(t); \quad \mathbf{S}_{\mathbf{x}}(t_{\mathrm{f}}) = \Phi^*_{\mathbf{xx}} \\
\dot{\mathbf{S}}_{\boldsymbol{\theta}}(t) &= -H^*_{\mathbf{x}\boldsymbol{\theta}} - \mathbf{S}_{\mathbf{x}}(t) \mathbf{F}^*_{\boldsymbol{\theta}} - {\mathbf{F}^*_{\mathbf{x}}}^{\mathrm{T}} \mathbf{S}_{\boldsymbol{\theta}}(t) && (23) \\
&\quad + \left[H^*_{\mathbf{xu}} + \mathbf{S}_{\mathbf{x}}(t) \mathbf{F}^*_{\mathbf{u}} \right] \mathbf{K}_{\boldsymbol{\theta}}(t); \quad \mathbf{S}_{\boldsymbol{\theta}}(t_{\mathrm{f}}) = \mathbf{0}.
\end{aligned}$$

Because this control law drives the first-variation of the NCO to zero, (19)–(23) can be seen as a NCO-tracking controller. Note that, for the integration of the sweep matrices $\mathbf{S}_{\mathbf{x}}$, $\mathbf{S}_{\boldsymbol{\theta}}$ to remain stable, the gain coefficients $\gamma^0_{2k}, \gamma^1_{2k}$, $0 \leq k < p$, must not be too large. A tradeoff must therefore be sought, which is clearly problem dependent.

5. CASE STUDY [1]

To illustrate the design and performance of NCO-tracking controllers for singular control problems, a steered car with inertia and friction is considered (see Fig. 1). A model describing the motion of the car is as follows

$$\begin{aligned}
\dot{y} &= V \cos \psi; & y(0) &= y_0 && (24) \\
\dot{z} &= V \sin \psi; & z(0) &= z_0 && (25) \\
\dot{V} &= u_1 - \mu V; & V(0) &= 0 && (26) \\
\dot{\psi} &= V \tan \phi; & \psi(0) &= 0 && (27) \\
\dot{\phi} &= u_2; & \phi(0) &= 0, && (28)
\end{aligned}$$

where y and z denote the Cartesian coordinates of the car [m], V its velocity [m/s], ψ its heading angle [rad], ϕ the orientation of its driving wheels [rad], and μ is a friction parameter [1/s]. The control variable u_1 represents the motor force divided by the mass of the car [N/kg], and the control variable u_2 stands for the rate of change of the orientation of the driving wheels [rad/s]. A quadrature variable, E, representing the cumulated energy consumption per unit mass [J/kg] is added to the model:

$$\dot{E} = u_1 V; \qquad E(0) = 0. \tag{29}$$

Note that $\dot{E}$ may be negative when u_1 opposes V, meaning that the car has the ability to recover energy from braking (regenerative braking system). The numerical values for the parameters, initial conditions, and input bounds are given in Tab. 1.

The optimization problem consists in minimizing the energy needed to bring the car to a neighborhood of the origin $(0, 0)$ at a fixed final t_{f}, while respecting input bounds:

[1] **Note to the reviewers.** As far as possible, the case study of a multiple-input chemical process will be included in the final version of the paper.

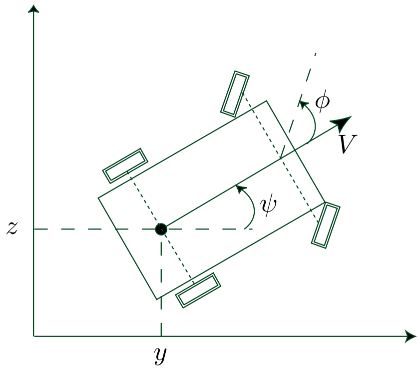

Fig. 1. Steered car with indication of the state variables.

$$\begin{aligned} \text{minimize :} \quad & J[\mathbf{u}] := y(t_f)^2 + z(t_f)^2 + E(t_f) & (30)\\ \text{subject to :} \quad & \text{model equations } (24) - (29) & (31)\\ & u_1^L \le u_1(t) \le u_1^U & (32)\\ & u_2^L \le u_2(t) \le u_2^U . & (33) \end{aligned}$$

5.1 Design of the NCO-Tracking Controller

The design of the NCO-tracking controller starts with the computation of the nominal optimal control. Possible values that can be taken by the inputs $u_1(t)$ and $u_2(t)$ along an optimal solution are determined upon application of Pontryagin's Maximum Principle:

- the interior arcs for the input u_1 are singular of degree $p_1 = 1$, and the values taken by $u_1^*(t)$ along an optimal solution are restricted to $\{u_1^L, u_1^U, \mu V(t) - \frac{\lambda_4(t)}{2\mu(\cos\phi(t))^2} u_2^L, \mu V(t) - \frac{\lambda_4(t)}{2\mu(\cos\phi(t))^2} u_2^U, \mu V(t)\}$;
- the interior arcs for the input u_2 are singular of degree $p_2 = 2$, and $u_2^*(t)$ can only take on discrete values in $\{u_2^L, u_2^U, 0\}$ along an optimal solution.

To get an idea of the optimal sequence of arcs, a numerical solution to the problem (30)–(33) is computed using the control vector parameterization (CVP) approach. The optimal solution appears to be constituted of 6 principal arcs, separated by 5 switching times t_k^*, $k = 1, \ldots, 5$ (Tab. 2).

Based on this arc sequence, a NCO-tracking controller is designed following the procedure described in Sections 3 and 4. In this controller, the junction times are fixed to

Table 1. Model parameters, initial conditions, and input bounds.

μ	0.1	1/s	u_1^L	−1	N/kg
y_0	−4	m	u_1^U	1	N/kg
z_0	4	m	u_2^L	−1	N/kg
t_f	8	s	u_2^U	1	rad/s

Table 2. Nominal arc sequence.

Arc	u_1	u_2
1	u_1^U	u_2^L
2	$u_1^{\text{sing}}(t)$	u_2^L
3	$u_1^{\text{sing}}(t)$	u_2^U
4	$u_1^{\text{sing}}(t)$	u_2^L
5	$u_1^{\text{sing}}(t)$	$u_2^{\text{sing}}(t)$
6	u_1^L	$u_2^{\text{sing}}(t)$

Table 3. NCO-tracking controller.

Arc	Controller for u_1	Controller for u_2
1	none	none
2	NE controller 1	none
3	NE controller 1	none
4	NE controller 1	none
5	NE controller 2	NE controller 2
6	none	NE controller 3

their nominal values t_k^*, $k = 1, \ldots, 5$, and the corrective actions taken along each arc are those indicated in Tab. 3.

The design of the three NE controllers is based on (19)–(23) and requires the computation of the time-varying matrices $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$. Although the SVD procedure described in Section 3 gives constant singular and nonsingular directions in this example (i.e., $\mathbf{U}_{2k}^{\text{ns}}$, $\mathbf{U}_{2k}^{\text{s}}$, $\mathbf{V}_{2k}^{\text{ns}}$, $\mathbf{V}_{2k}^{\text{s}}$ are constant matrices), performing those calculations by hand is both fastidious and prone to errors. Fortunately, the theory lends itself naturally to automation, e.g., with the symbolic toolbox of MATLAB®.

- In the NE controller 1, the recursion starts with the conditions
$$\mathbf{0} = \delta\boldsymbol{\mathcal{L}}(t) = \begin{pmatrix} \delta\lambda_3(t) + \delta V(t) \\ \delta u_2(t) \end{pmatrix}, \quad t_1^* < t \le t_4^*.$$
The matrix $\mathbf{C}_0$ has rank $n_u - r_0 = 1$, and 2 time differentiations are required to devise the controller, i.e. $p = 1$.
- In the NE controller 2, the recursion starts with the conditions
$$\mathbf{0} = \delta\boldsymbol{\mathcal{L}}(t) = \begin{pmatrix} \delta\lambda_3(t) + \delta V(t) \\ \delta\lambda_5(t) \end{pmatrix}, \quad t_4^* < t \le t_5^*.$$
The matrix $\mathbf{C_0}$ has rank $n_u - r_0 = 0$. The computations give $r_2 = 1$ after 2 rounds of differentiations, and a total of 4 successive differentiations is necessary to design the controller, i.e. $p = 2$.
- Finally, the recursion for the NE controller 3 starts with the conditions
$$\mathbf{0} = \delta\boldsymbol{\mathcal{L}}(t) = \begin{pmatrix} \delta u_1(t) \\ \delta\lambda_5(t) \end{pmatrix}, \quad t_5^* < t \le t_f.$$
The matrix $\mathbf{C_0}$ has rank $n_u - r_0 = 1$. The computations give $r_2 = 1$ after 2 rounds of differentiations, and a total of 4 successive differentiations is necessary to design the controller, i.e. $p = 2$.

The gain coefficients in the NE controller 1 are taken as $\gamma_0^0 = s_1 s_2$ and $\gamma_0^1 = -s_1 s_2$, so that the differential operator T_0 given in (17) has stable poles at $s_1 = -0.2$ and $s_2 = -2$. On the other hand, the gain coefficients in the NE controllers 2 and 3 are taken as $\gamma_0^0 = \gamma_2^0 = s^2$ and $\gamma_0^1 = \gamma_2^1 = -2s$, so that T_0 and T_2 both have stable poles at $s = -1$. Decreasing the poles leads to more aggressive NE controllers in the sense that the inputs are more likely to saturate at their upper or lower bounds. Generally, decreasing the poles also has an adverse effect on the stability of the backward sweep integration.

5.2 Performance of the NCO-Tracking Controller

To assess the performance of the proposed control approach, a scenario is considered where the initial conditions are perturbed as $\psi(0) = -0.175$ [rad] and $V(0) = 0.5$

Table 4. Compared performance of various control strategies.

Control Strategy	J
Optimal Solution (Ideal)	0.7015
Open-Loop Nominal Solution	10.084
NCO Tracking	0.8034
NCO Tracking with NE Control on Arc 5 only	1.1005

[m/s], and the friction parameter is perturbed as $\mu = 0.2$ [1/s]. Note that these are substantial perturbations of the nominal operating conditions.

The NCO-tracking controller is compared to:

(1) the optimal solution to the perturbed problem, assuming known perturbations;
(2) the optimal solution to the nominal problem, applied open loop;
(3) a simplified NCO-tracking controller implementing a two-input NE controller on Arc 5 and applying the nominal control open loop along the remaining arcs.

The input and response trajectories for the various control strategies are shown in Fig. 2. Moreover, the cost obtained with each strategy is reported in Tab. 4. Note that applying the nominal input open loop leads to poor performance. On the other hand, the proposed NCO-tracking controller is able to recover most of the performance loss compared to the optimal strategy (which assumes that the perturbed initial conditions are known). Interestingly enough, most of the performance loss is also recovered upon application of a two-input NE controller along Arc 5 only (NE controller 2), while leaving the remaining arcs uncontrolled. This comparison clearly illustrates the central role played by the NE controller 2 in the NCO-tracking scheme.

6. CONCLUSIONS AND FUTURE WORK

Many practical problems of interest exhibit solutions that contain singular arcs, e.g., in rocket and air vehicle flight or chemical plant operation. Strong incentives therefore exist to operate these processes in the most efficient possible manner, despite the presence of uncertainty. In this paper, an extension of the theory of neighboring extremals to address singular optimal control problem has been proposed for multiple-input system. Moreover, to make these results tractable from a real-time optimization perspective, an explicit NCO-tracking controller has been devised, which guarantees that, under mild assumptions, the first-order variation of the NCO converge to zero exponentially. These results have been illustrated by the case study of a steered car. It is found that the NCO-tracking controller not only allows to recover most of the optimality loss induced by large perturbations of the initial conditions, but it also shows excellent performance in the presence of substantial parameter uncertainty.

Despite the apparent complexity of designing NE controllers for singular control problems, it should be emphasized that most of this complexity is dealt with off-line. In particular, on-line calculations are limited to the implementation of a simple state-feedback law. This approach is therefore well suited to control fast dynamical systems. An important limitation of the approach lies in the fact that full state measurement is required in the NE feedback law. The design of NE controllers based on output feedback will be the topic of future research. Future developments will also aim at handling those optimal control problems for which fixing the junction leads to large performance loss.

REFERENCES

D. J. Bell and D. H. Jacobson. *Singular Optimal Control Problems.* Academic Press, 1975.

A. E. Bryson and Y. C. Ho. *Applied Optimal Control.* Hemisphere, Washington DC, 1975.

S. Gros. *Neighboring Extremals in Optimization and Control.* PhD thesis, École Polytechnique Fédérale de Lausanne, Switzerland, 2007.

S. Gros, B. Srinivasan, and D. Bonvin. Neighboring extremal controllers for singular problems. In *Proc. 2004 American Control Conference*, pages 34–39, Boston, MA, 2004.

L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes.* Pergamon Press, New York, 1964.

H. M. Robbins. A generalized legendre-clebsch condition for the singular cases of optimal control. *IBM J Res Dev*, 11(4):361–372, 1967.

B. Srinivasan and D. Bonvin. Real-time optimization of batch processes via tracking of necessary conditions of optimality. *Ind Eng Chem Res*, 46(2):492–504, 2007.

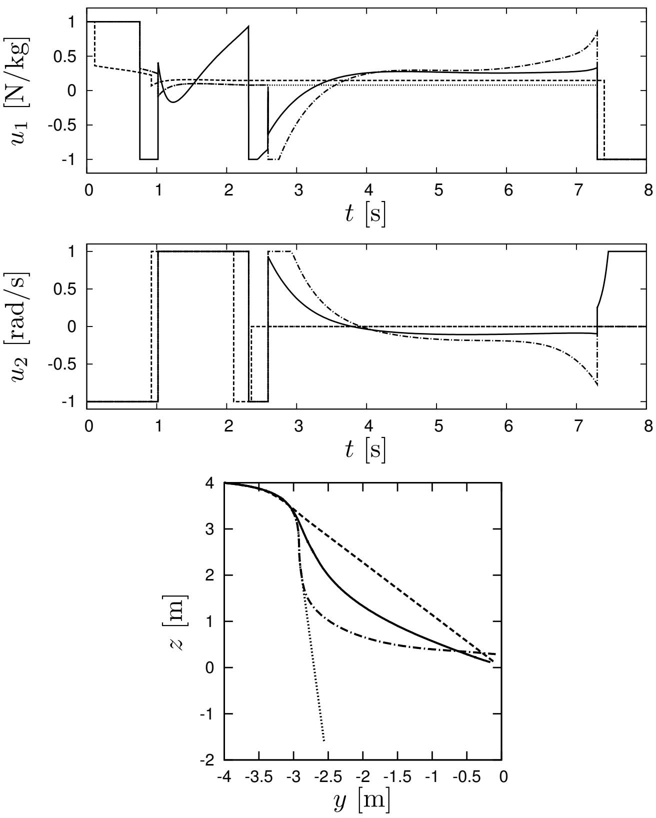

Fig. 2. Comparison of the input and state trajectories obtained with different control strategies. Dashed line: optimal solution to the perturbed problem; solid line: NCO tracking; dash-dotted line: NCO tracking on Arc 5 only; dotted line: optimal nominal solution applied open loop.