Reduced basis approximation of parametrized advection-diffusion PDEs with high Péclet number

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Abstract. In this work we show some results about the reduced basis approximation of advection dominated parametrized problems, i.e. advection-diffusion problems with high Péclet number. These problems are of great importance in several engineering applications and it is well known that their numerical approximation can be affected by instability phenomena. In this work we compare two possible stabilization strategies in the framework of the reduced basis method, by showing numerical results obtained for a steady advection-diffusion problem.

1 Introduction

We show here some recent results about *stabilized reduced basis methods* for the approximation of parametrized advection-diffusion problems with high Péclet number, which expresses the ratio between the advection term and the diffusion one.

Advection-diffusion problems are effectively employed to model a wide range of physical phenomena. Just to give an example, we can recall heat transfer phenomena (with conducion and convection) [11] or diffusion of pollutants in the atmosphere [2,9]. These equations can depend on several parameters, typically the Péclet number, the advection field direction and the geometry of the domain.

Moreover, parametrized advection-diffusion equations are often used in engineering applications which require very fast evaluations of the solution, given particular values of the parameters. The reduced basis method [8,12] can effectively provide a *rapid* approximation of the solution, as well as *rigorous* error bounds, which guarantee the *reliability* of the solution. A very important feature of the reduced basis method is its decomposition in two computational stages. In the first expensive stage, called *Offline* stage, some high-fidelity solutions of the problems are computed, which will become the basis functions for the Galerkin projection performed in the second inexpensive stage, called *Online* stage.

Some applications of the reduced basis method to advection-diffusion problems, such as the Graetz problem or the "thermal fin" problem, can be found in literature, especially the case in which the Péclet number is moderate (i.e. $\sim 10^2$) [3,5,8,11,13].

When the Péclet number takes higher values, the finite elements (FE) approximation of advection-diffusion problems can show significant instability phenomena (see e.g. [10]). To overcome this problem, one can resort to some classical stabilization method, like the *Streamline/Upwind Petrov Galerkin* (SUPG) method [1]. In this way, it is possible to compute a stable approximated solution suitable to be considered as the *truth* one, i.e, the reference high-fidelity solution for the RB method. A first investigation of the coupling between the stabilized FE formulation and the RB method has been done in [2,9]. We now base our work on some more recent results given in [6]. Following the latter work, we want to compare two possible strategies of stabilization, by comparing some numerical results in the steady case. The first one, that we will call *Offline-online stabilized* method consists in "stabilize" both the Offline and the Online stages, i.e. using the same stabilized bilinear form in both stages. This method has been actually applied in [2,9]. The other method, called *Offline-only* consists in "stabilizing" only the Offline stage and then perform the Online stage using the standard advection-diffusion operator. To explain the underlying idea, first of all we recall that the RB solution is actually a linear combination of few reduced basis (i.e. the high-fidelity solutions computed during the Offline stage) [12]. It can then be reasonable to expect that if our reduced basis are stable, the reduced solution obtained using the non-stabilized advection-diffusion operator will be stable too. After this brief introduction, in Section 2 we recall the stabilized reduced basis method, in Section 3 we show some numerical tests and, finally, in Section 4 we draw some conclusions.

2 Stabilized reduced basis method

We take now into account a general parametric advection diffusion problem:

$$-\varepsilon(\boldsymbol{\mu})\Delta u(\boldsymbol{\mu}) + \boldsymbol{\beta}(\boldsymbol{\mu}) \cdot \nabla u(\boldsymbol{\mu}) = 0 \quad \text{on } \Omega. \tag{1}$$

given a parameter value $\boldsymbol{\mu}$ in the parameter domain $\mathcal{D}$ and suitably chosen Dirichlet, Neumann or mixed boundary conditions. We consider a domain Ω which is an open subset of $\mathbb{R}^2$. As regards the coefficients, we consider sufficiently regular functions $\varepsilon(\boldsymbol{\mu})\colon \Omega \to \mathbb{R}$ and $\boldsymbol{\beta}(\boldsymbol{\mu})\colon \Omega \to \mathbb{R}^2$. The bilinear form associated with the advection-diffusion problem is:

$$a(u, v; \boldsymbol{\mu}) = \int_\Omega \varepsilon(\boldsymbol{\mu})\nabla u \cdot \nabla v + \boldsymbol{\beta}(\boldsymbol{\mu}) \cdot \nabla u\, v \qquad \forall u, v \in H^1(\Omega). \tag{2}$$

Given a triangulation $\mathcal{T}_h$ defined on Ω, with maximum element diameter h, we can set up a FE approximation of the advection-diffusion problem [7].

We denote with $X^{\mathcal{N}}$ the space of piecewise-linear finite elements. It is very well known in literature (see e.g. [7,10]) that the FE approximation can show instability phenomena when the advective terms dominates the diffusive one. More precisely, we say that a problem is *advection dominated in* $K \subset \Omega$ if the following condition holds:

$$\mathbb{P}\mathsf{e}_K(\boldsymbol{\mu})(x) := \frac{|\boldsymbol{\beta}(\boldsymbol{\mu})(x)|h_K}{2\varepsilon(\boldsymbol{\mu})(x)} > 1 \quad \forall \boldsymbol{x} \in K \quad \forall \boldsymbol{\mu} \in \mathcal{D}, \tag{3}$$

where h_K is the diameter of K.

In order to obtain an approximated solution which does not show instabilities, we can resort to some stabilization method. We decided to exploit the classical SUPG method [1]. This consists in substituting, in the FE formulation, the standard advection-diffusion bilinear form (2) with the following one

$$\begin{aligned} a_{stab}(w^{\mathcal{N}}, v^{\mathcal{N}}; \boldsymbol{\mu}) = &\int_{\Omega} \varepsilon(\boldsymbol{\mu}) \nabla w^{\mathcal{N}} \cdot \nabla v^{\mathcal{N}} + (\boldsymbol{\beta}(\boldsymbol{\mu}) \cdot \nabla w^{\mathcal{N}}) v^{\mathcal{N}} \\ &+ \sum_{K \in \mathcal{T}_h} \delta_K \int_K L^{\boldsymbol{\mu}} v^{\mathcal{N}} \left(\frac{h_K}{|\boldsymbol{\beta}(\boldsymbol{\mu})|} L^{\boldsymbol{\mu}}_{SS} v^{\mathcal{N}} \right) \end{aligned} \tag{4}$$

with $w^{\mathcal{N}}, v^{\mathcal{N}}$ chosen in $X^{\mathcal{N}}$. In (4) $L^{\boldsymbol{\mu}}$ is the advection-diffusion operator $L^{\boldsymbol{\mu}} v^{\mathcal{N}} = -\varepsilon(\boldsymbol{\mu}) \Delta v^{\mathcal{N}} + \boldsymbol{\beta}(\boldsymbol{\mu}) \cdot \nabla v^{\mathcal{N}}$, while $L^{\boldsymbol{\mu}}_{SS}$ is its skew-symmetric part. Note that in the case of a divergence free advection field $\boldsymbol{\beta}(\boldsymbol{\mu})$, it holds that $L^{\boldsymbol{\mu}}_{SS} = \boldsymbol{\beta}(\boldsymbol{\mu}) \cdot \nabla v^{\mathcal{N}}$ [10]. The weights δ_K have to be properly chosen in order to ensure the stability and convergence of the SUPG method [7,10].

We can now consider the RB approximation of the problem (1). As regards the Offline stage, we decided to consider only the stabilized bilinear form (4) and thus we considered as *truth* solution the SUPG stabilized one, that is to find $u^{s\mathcal{N}}(\boldsymbol{\mu}) \in X^{\mathcal{N}}$ such that

$$a_{stab}(u^{s\mathcal{N}}, v^{\mathcal{N}}; \boldsymbol{\mu}) = f_{stab}(v^{\mathcal{N}}; \boldsymbol{\mu}) \quad \forall\, v^{\mathcal{N}} \in X^{\mathcal{N}}. \tag{5}$$

where the right-hand side functional f can be a forcing term or can depend on the imposition of boundary conditions. Considering problem (5), we can set up the Offline stage of the RB method, which produces a reduced space $X^{\mathcal{N}}_N \subset X^{\mathcal{N}}$ with dimension N such that $N \ll \mathcal{N}$.

For the Online stage, we propose two different strategies. The first one, which correspond to the *Offline-Online stabilized* method, consists in using the stabilized bilinear form also during the Online stage. Then the Online problem turns out to be: find $u^s_N(\boldsymbol{\mu}) \in X^{\mathcal{N}}_N$ such that

$$a_{stab}(u^s_N(\boldsymbol{\mu}), v_N; \boldsymbol{\mu}) = f_{stab}(v_N; \boldsymbol{\mu}) \quad \forall\, v_N \in X^{\mathcal{N}}_N. \tag{6}$$

On the contrary in the second method we propose, the *Offline-only stabilized* method, the Online stage is performed using the original advection-diffusion bilinear form (2). The Online problem is then: find $u_N(\boldsymbol{\mu}) \in X^{\mathcal{N}}_N$ such that

$$a(u_N(\boldsymbol{\mu}, v_N; \boldsymbol{\mu}) = f(v_N; \boldsymbol{\mu}) \quad \forall\, v_N \in X^{\mathcal{N}}_N. \tag{7}$$

The right-hand side functional f can in general be different from the one of the stabilized problem, because it does not contain, for example, contributions given by the stabilization term and the lifting of Dirichlet boundary conditions.

3 Numerical test: advection-diffusion problem with a boundary layer.

We consider now the following advection-diffusion problem, whose domain $\Omega_o(\boldsymbol{\mu})$ is sketched in Fig. 1,

$$\begin{cases} -\frac{1}{\mu_1}\Delta u(\boldsymbol{\mu}) + \boldsymbol{\beta}\cdot\nabla u(\boldsymbol{\mu}) = 0 & \text{in } \Omega_o(\boldsymbol{\mu}) \\ u(\boldsymbol{\mu}) = 0 & \text{on } \Gamma_{o,1}(\boldsymbol{\mu}) \cup \Gamma_{o,2}(\boldsymbol{\mu}) \\ \frac{1}{\mu_1}\frac{\partial u}{\partial n}(\boldsymbol{\mu}) = 0 & \text{on } \Gamma_{o,3}(\boldsymbol{\mu}) \\ \frac{1}{\mu_1}\frac{\partial u}{\partial n}(\boldsymbol{\mu}) = 1 & \text{on } \Gamma_{o,4}(\boldsymbol{\mu}) \end{cases} \tag{8}$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2)$ belongs to $\mathcal{D} = [100, 1000] \times [2, 6]$. We choose $\boldsymbol{\beta} = (y, -0.1)$. In order to effectively perform a RB approach, we need to choose a reference domain Ω, as described [8,12]. We thus set $\Omega = \Omega_o(\mu_2 = 3)$ on which we define the FE triangulation. We also define an affine transformation $T^{\boldsymbol{\mu}} \colon \Omega \to \Omega_o(\boldsymbol{\mu})$ which maps the reference domain onto the parametrized one, which is $T^{\boldsymbol{\mu}}(x, y) = (\mu_2\, x/3, y)$.

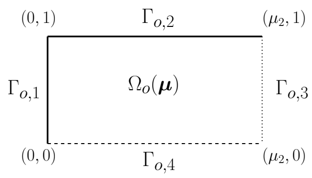


Fig. 1: Domain of problem (8). The boundary conditions are: homogeneous Dirichlet on the bold sides, homogeneous Neumann on the dotted side and non-homogeneous Neumann on the dashed side.

Using the transformation $T^{\boldsymbol{\mu}}$ we can track back to the reference domain all the bilinear forms defined on the parametrized domain. The transformed advection-diffusion bilinear forms turns out to be:

$$\begin{aligned} a(w^{\mathcal{N}}, v^{\mathcal{N}}; \mu) = \frac{3}{\mu_1\,\mu_2}\int_\Omega \partial_x w^{\mathcal{N}} \partial_x v^{\mathcal{N}} + \frac{\mu_2}{3\,\mu_1}\int_\Omega \partial_y w^{\mathcal{N}} \partial_y v^{\mathcal{N}} \\ + \int_\Omega y\,\partial_x w^{\mathcal{N}}\, v^{\mathcal{N}} - \frac{\mu_2}{30}\int_\Omega \partial_y w^{\mathcal{N}}\, v^{\mathcal{N}}, \end{aligned} \tag{9}$$

for all $w^{\mathcal{N}}$, $v^{\mathcal{N}}$ in $X^{\mathcal{N}}$. Note that the bilinear form (9), satisfies the *affinity* assumption

$$a(w^{\mathcal{N}}, v^{\mathcal{N}}; \boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu}) a^q(w^{\mathcal{N}}, v^{\mathcal{N}}) \quad \forall \boldsymbol{\mu} \in \mathcal{D}, \tag{10}$$

where Θ_a^q, $q = 1, \ldots, Q_a$, are functions $\mathcal{D} \to \mathbb{R}$ while a^q, $q = 1, \ldots, Q_a$, are $\boldsymbol{\mu}$-independent bilinear forms on $X^{\mathcal{N}}$. Assumption (10) is crucial for the efficiency of the Offline/Online decomposition of the RB method [8,12].

As regards the stabilization term, we point out that for piecewise linear approximation we do not have particular restriction on the choice of the weights δ_K [10]. We then set $\delta_K = 1$ for each element K. As piecewise linear functions have null Laplacian inside each element, the stabilization term becomes:

$$\begin{aligned} s(w^{\mathcal{N}}, v^{\mathcal{N}}; \boldsymbol{\mu}) = &\sqrt{\frac{1+\mu_2^2}{10}} \left[\frac{3}{\mu_2} \sum_{K \in \mathcal{T}_h} h_K \int_K y^2 \partial_x w^{\mathcal{N}} \partial_x v^{\mathcal{N}} \right. \\ &+ \sum_{K \in \mathcal{T}_h} h_K \int_K 2\, y \,(\partial_x w^{\mathcal{N}} \partial_y v^{\mathcal{N}} + \partial_y w^{\mathcal{N}} \partial_x v^{\mathcal{N}}) \\ &\left. + \frac{\mu_2}{3} \sum_{K \in \mathcal{T}_h} h_K \int_K \partial_y w^{\mathcal{N}} \partial_y v^{\mathcal{N}} \right]. \end{aligned} \tag{11}$$

for all $w^{\mathcal{N}}$, $v^{\mathcal{N}}$ in $X^{\mathcal{N}}$. The term $\sqrt{(1+\mu_2^2)/10}$ has been inserted to keep into account the transformation of the element diameter. In order to ensure the *affinity* assumption (10) also for the stabilization term, with $Q_a \ll \mathcal{N}$, we assumed that each element diameter transforms as the diameter of the whole domain. Considering the exact transformation for each element diameter would have implied a number of affine terms of the order of $\mathcal{N}$ (one affine term per element).

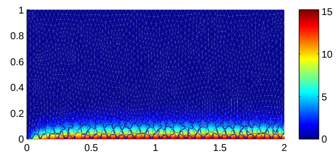
Having defined forms (9) and (11), we can define the stabilized bilinear form $a_{stab} = a + s$. Now we can set up the Offline stage of the RB method, to be performed with respect to the stabilized bilinear form a_{stab}. We applied the Successive Constraint Method (SCM) [4,12] to build computationally inexpensive lower bounds for the parametric coercivity constants and then we applied the standard RB Greedy algorithm [8,12].

In our computations, the Offline stage required 311 s (237 s for the SCM) and produced a reduced space with $N = 26$ basis. The tolerance on the Greedy algorithm was $\varepsilon_{tol}^* = 10^{-3}$. This means that we can guarantee that

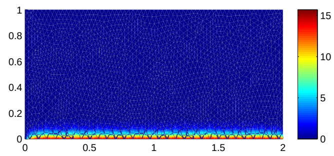
$$|||u_N^s(\boldsymbol{\mu}) - u^{s\,\mathcal{N}}(\boldsymbol{\mu})|||_{\boldsymbol{\mu},stab} \leq \varepsilon_{tol}^* \quad \forall \boldsymbol{\mu} \in \Xi \tag{12}$$

where Ξ is a sufficiently large subset of $\mathcal{D}$ with finite cardinality (see [12]). In (12), $|||\cdot|||_{\boldsymbol{\mu},stab}$ is the norm induced by the symmetric part of the bilinear form a_{stab}.

We can now compare the *Offline-Online stabilized* method and the *Offline-only stabilized* method. In Fig. 2 we show some *Offline-Online* approximated solutions, while in Fig. 3 we show some *Offline-only* approximated solutions. It is evident that the solutions produced with the *Offline-only stabilized* method can show significant instabilities, as shown in Fig. 3b. We have actually shown that *a Galerkin projection on a subspace spanned by stable functions does not guarantee that the solution does not show instability phenomena.* On the contrary, we observe that the *Offline-Online stabilized* method always produces stable solutions.

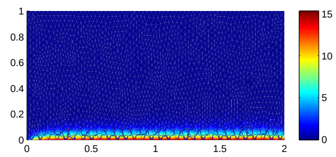


(a) $\boldsymbol{\mu} = (200, 3)$

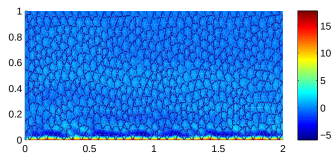


(b) $\boldsymbol{\mu} = (900, 3)$

Fig. 2: *Offline-Online stabilized* method. Solutions for some representative values of the parameter.



(a) $\boldsymbol{\mu} = (200, 3)$



(b) $\boldsymbol{\mu} = (900, 3)$

Fig. 3: *Offline-only stabilized* method. Solutions for some representative values of the parameter.

In order to understand the bad behaviour of the *Offline-only stabilized* method for our problem, the following upper bound can been proven using the same arguments of [6],

Proposition 1 (Upper bound for the *Offline-only* method). *The following estimate of the error between the Offline-only stabilized approximation* $u_N(\boldsymbol{\mu})$ *and the stabilized FE approximation* $u^{s\mathcal{N}}(\boldsymbol{\mu})$ *holds:*

$$\begin{aligned} |||u_N(\boldsymbol{\mu}) - u^{s\mathcal{N}}(\boldsymbol{\mu})|||_{\boldsymbol{\mu}} \leq \; & |||u^s_N(\boldsymbol{\mu}) - u^{s\mathcal{N}}(\boldsymbol{\mu})||| \\ & + h_{max}\sqrt{\mu_1 \frac{1+\mu_2^2}{10}} \|\boldsymbol{\beta} \cdot \nabla(u^s_N(\boldsymbol{\mu}) + g_h)\|_{L^2(\Omega_o(\boldsymbol{\mu}))} \end{aligned} \tag{13}$$

where $u_N^s(\boldsymbol{\mu})$ *is the Offline-Online stabilized solution,* g_h *is the lifting of the Dirichlet boundary condition and* $|||\cdot|||_{\boldsymbol{\mu}}$ *is the norm induced by the symmetric part of the bilinear form* a*. The value* h_{max} *is the maximum element diameter of the reference mesh* $\mathcal{T}_h$*.*

In Fig. 4 we show a comparison between the *Offline-Online* approximation error, the *Offline-Online* approximation error and the upper bound (13), having fixed $\mu_2 = 3$. The reasonable sharpness shown by the upper bound suggests that in general the *Offline-Only stabilized* method is not a good approximation strategy. We can also highlight that a major component of the *Offline-only* error can be the streamline derivative term in (13). This is also suggested by the fact that, when the streamline derivative term in (13) is "small", e.g. when the advection field and the boundary layer are almost parallel and both the advection field and the gradient of the solution have relatively small modulus, then the *Offline-only stabilized* method can produce satisfactory results too, as shown in [6] for a Graetz problem.

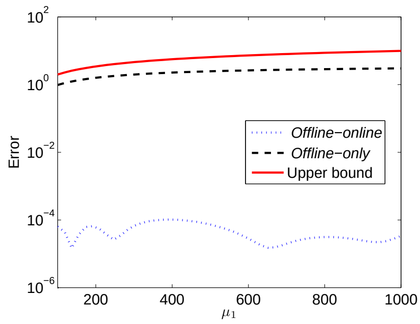


Fig. 4: Approximation errors and upper bound as functions of μ_1, for $\mu_2 = 3$ fixed.

4 Conclusions

We have investigated the RB approximation of advection dominated RB problems, comparing two possible strategies an *Offline-Online stabilized* method and an *Offline-only stabilized* one. Numerical results have shown that the former gives better results, while the latter produces reduced solutions with strong instability effects, even if the reduced basis functions are stable. We have shown that the numerical results obtained are in accordance with the theoretical estimates proven in [6].

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