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DIELECTRIC TENSOR OPERATOR OF HOT PLASMAS IN TOROIDAL AXISYMMETRIC SYSTEMS

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Dielectric Tensor Operator of Hot Plasmas in Toroidal Axisymmetric Systems

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Abstract

Kinetic theory is used to develop equations describing dynamics of small-amplitude electromagnetic perturbations in toroidal axisymmetric plasmas. The closed Vlasov-Maxwell equations are first solved for a hot stationary plasma using the expansion in the small parameter $\epsilon_e = \rho/L$, where ρ is the Larmor radius and L a characteristic length scale of the stationary state. The ordering and additional assumptions are specified so as to obtain the well-known Grad-Shafranov equation. The dielectric tensor of such a plasma is then derived. The Vlasov equation for the perturbed distribution function is solved by the expansion in the small parameters ϵ_e and $\epsilon_p = \rho/\lambda$, where λ is a characteristic wavelength of the perturbing electromagnetic field. The solution is obtained up to the first order in ϵ_e and the second order in ϵ_p. By integrating the resulting distribution function over velocity space, an explicit expression for the tensor is derived in the form of a two-dimensional partial differential operator. The operator is shown to possess the proper symmetry corresponding to the energy conservation law.

I. Introduction

The dielectric tensor operator of hot inhomogeneous plasmas has recently been derived for slab[1-3] and cylindrical[4] geometries. The objective of this paper is to obtain the corresponding tensor for a toroidal axisymmetric system. Implemented in a numerical code, it will provide a tool for studying electromagnetic wave propagation and instabilities in tokamak-like plasmas. The calculation is carried out under the following assumptions :

- The unperturbed system is a stationary toroidal axisymmetric plasma .
- The hot plasma is assumed to be collisionless so that each species is described by a distribution function obeying the Vlasov equation.
- The two parameters $\epsilon_e = \rho/L$ and $\epsilon_p = \rho/\lambda_\perp$ are considered small, where ρ is the Larmor radius, L a characteristic length scale of the stationary plasma and $\lambda_\perp$ a characteristic wavelength of the perturbing electromagnetic field perpendicular to the toroidal direction (that is the field in the poloidal plane). The stationary state is considered up to the first order in ϵ_e, the perturbation by the electromagnetic field is evaluated up to the orders $\epsilon_e\epsilon_p$and ϵ_p^2.
- To the zeroth order in ϵ_e the stationary plasma is in a local equilibrium, the distribution function is therefore local Maxwellian.

As for any linear response calculation, the unperturbed state must first be considered. This is done in Sec.II. The solution of the linearized Vlasov equation to the order desired is then obtained in Sec.III while the derivation of the dielectric tensor itself is presented in Sec.IV. Finally we draw conclusions in Sec.V.

II. The Stationary State

The distribution function $f(\vec{x}, \vec{v}, t)$ of each species of mass m and charge q obeys the Vlasov equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{q}{m} \left[\vec{E} + \vec{v} \times (\vec{B} + \vec{\tilde{B}}) \right] \cdot \frac{\partial f}{\partial \vec{v}} = 0, \tag{1}$$

where $\vec{B}$ and $(\vec{E}, \vec{\tilde{B}})$ are the magnetostatic and electromagnetic fields obeying the Maxwell equations. The stationary distribution function f_0 of the unperturbed plasma therefore satisfies

$$\vec{v} \cdot \frac{\partial f_0}{\partial \vec{x}} + \Omega(\vec{v} \times \vec{e}_{\|}) \cdot \frac{\partial f_0}{\partial \vec{v}} = 0, \tag{2}$$

where $\Omega = qB/m$ is the cyclotron frequency and $(\vec{e}_n, \vec{e}_b, \vec{e}_{\|})$ an orthonormal magnetic coordinate system such that

$$\vec{e}_{\|} = \frac{\vec{B}}{B} \qquad \text{and} \qquad \vec{e}_n \times \vec{e}_b = \vec{e}_{\|} \, . \tag{3}$$

Knowing that for a charged particle in a magnetostatic field its energy is an invariant and its magnetic moment an adiabatic invariant, it is convenient to adopt the gyrokinetic variables $(\vec{x}, \epsilon, \mu, \alpha, \sigma)$ where

$$\vec{v} = v_{\perp}(\cos\alpha \; \vec{e}_n + \sin\alpha \; \vec{e}_b) + v_{\|} \; \vec{e}_{\|}, \tag{4}$$

$$\epsilon = \frac{v^2}{2}, \qquad \mu = \frac{v_{\perp}^2}{2B}, \qquad \sigma = \mathrm{sgn}(v_{\|}). \tag{5}$$

Equation (2) is thus transformed to

$$\left[\mathcal{L}^{(1)} - \frac{\partial}{\partial \alpha} \right] f_0 = 0 \tag{6}$$

with

$$\begin{aligned} \mathcal{L}^{(1)} \;=\; & \frac{1}{\Omega} \Big\{ \vec{v} \cdot \nabla - \frac{1}{B} \left[v_{\|} \, \vec{v} \cdot (\nabla \, \vec{e}_{\|}) \cdot \vec{v} + \mu \, \vec{v} \cdot (\nabla B) \right] \frac{\partial}{\partial \mu} \\ & + \left[\vec{v} \cdot (\nabla \, \vec{e}_b) \cdot \vec{e}_n + \frac{v_{\|}}{v_{\perp}^2} \, \vec{v} \cdot (\nabla \, \vec{e}_{\|}) \cdot (\vec{v}_{\perp} \times \vec{e}_{\|}) \right] \frac{\partial}{\partial \alpha} \Big\}. \end{aligned} \tag{7}$$

Notice that the geometry of the magnetic field is characterized by the coefficients

$$T_{\mu\nu\sigma} = \vec{e}_\mu \cdot (\nabla \, \vec{e}_\nu) \cdot \vec{e}_\sigma \qquad\qquad \mu, \nu, \sigma \in \{n, b, \|\}. \tag{8}$$

These twenty-seven coefficients (Christoffel symbols) can be expressed in terms of the nine coefficients

$$\beta_{\mu\nu} = \vec{e}_\nu \cdot (\nabla \times \vec{e}_\mu) \qquad\qquad \mu, \nu \in \{n, b, \|\}, \tag{9}$$

using well-known formulae of vector analysis. One obtains the following relations

$$\begin{array}{lll} T_{nnb} = \beta_{n\|}, & T_{n\|n} = \beta_{nb}, & T_{n\|b} = \frac{1}{2}(\beta_{bb} + \beta_{\|\|} - \beta_{nn}), \\ T_{bnb} = \beta_{b\|}, & T_{bn\|} = \frac{1}{2}(\beta_{nn} + \beta_{\|\|} - \beta_{bb}), & T_{bb\|} = \beta_{bn}, \\ T_{\|bn} = \frac{1}{2}(\beta_{nn} + \beta_{bb} - \beta_{\|\|}), & T_{\|\|n} = \beta_{\|b}, & T_{\|b\|} = \beta_{\|n}, \end{array} \tag{10}$$

$$T_{\mu\nu\sigma} = -T_{\mu\sigma\nu}.$$

We shall solve Eq.(6) using a perturbation method. For this reason we write f_0 in a series expansion with respect to the small parameter $\epsilon_e = \rho/L$, which compares the variation of equilibrium quantities with the Larmor radius $\rho = v/\Omega$

$$f_0 = F + F^{(1)} + F^{(2)} + \ldots \tag{11}$$

Noticing that the operator $\mathcal{L}^{(1)}$ is of order one in ϵ_e, Eq.(6) written in zeroth order shows that F is independent of α

$$F \neq F(\alpha). \tag{12}$$

The fact that f_0 is periodic in α imposes a further constraint on F. This can be obtained by the annihilation method, which consists in writing Eq.(6) to first order in ϵ_e and averaging it with respect to α over the interval $[0, 2\pi]$. It yields

$$\frac{1}{2\pi} \int_0^{2\pi} d\alpha \, \mathcal{L}^{(1)} F \equiv < \mathcal{L}^{(1)} > F = 0, \tag{13}$$

which implies

$$\nabla_\| F = 0, \tag{14}$$

where relations (10) and the incompressibility of the magnetic field were used (we adopt the notation $\nabla_\nu = \vec{e}_\nu \cdot \nabla$).

Solving Eq.(6) to first order in ϵ_e gives

$$F^{(1)} = \left(\int^\alpha \mathcal{L}^{(1)} d\alpha \right) F + G^{(1)}, \tag{15}$$

$G^{(1)}$ being a function independent of α, which is yet to be determined. We shall obtain $G^{(1)}$ by applying the annihilation method to Eq.(6) written to second order. So as to render this procedure feasible, the distribution function at zero order must be specified. We assume that the plasma is close to a local equilibrium; F is therefore essentially Maxwellian. To create a necessary internal current a slight anisotropy must however subsist in the velocity distribution. As F is independent of α, the anisotropy is expressed by its dependence on σ, which allows for a current along the magnetic field, the so-called force-free current. We therefore set

$$\begin{aligned} F &= F_M(1+\delta^{(1)}), \\ F_M &= \frac{N}{(\pi v_{th}^2)^{\frac{3}{2}}} \exp(-2\frac{\epsilon}{v_{th}^2}), \qquad \delta^{(1)} = \delta^{(1)}(\vec{x}_\perp, \epsilon, \mu, \sigma), \end{aligned} \tag{16}$$

F_M being the Maxwellian distribution with the local density $N = N(\vec{x}_\perp)$ and the thermal speed squared $v_{th}(\vec{x}_\perp)^2 = 2T/m$, $\delta^{(1)}$ characterizing the anisotropy. We assume that the anisotropy is small and thus consider $\delta^{(1)}$ as a term of order one in ϵ_e. The first order contribution to the distribution function can then be written as

$$F^{(1)} = \frac{\vec{v}_\perp \times \vec{e}_\parallel}{\Omega} \cdot \nabla F_M + G^{(1)}. \tag{17}$$

Applying the annihilation method thus yields,

$$< \mathcal{L}^{(1)} F^{(1)} >= 0, \tag{18}$$

which implies

$$v_\parallel \nabla_\parallel G^{(1)} = \frac{1}{\Omega} \left[(v_\parallel^2 + \frac{1}{2} v_\perp^2)(\nabla \ln B) \cdot (\vec{e}_\parallel \times \nabla) - \mu \nabla \cdot (\vec{B} \times \nabla) \right] F_M, \tag{19}$$

where relations (10), Ampere's law and the fact that the current density is at least of order one in ϵ_e were used.

We now restrict ourselves to an axisymmetric system. In cylindrical coordinates (r,φ,z) all physical quantities are thus independent of the toroidal angle φ. Let us consider the magnetic flux through a surface enclosed by a toroidal line, given to a factor 2π by the function

$$\psi(r,z) = \frac{1}{2\pi} \oint_{\Gamma(r,z)} \vec{B} \cdot d\vec{\sigma} = rA_\varphi, \tag{20}$$

where A_φ is the toroidal component of the vector-potential ($\vec{B} = \nabla \times \vec{A}$). The magnetic field can then be written as follows

$$\vec{B} = \nabla\psi \times \nabla\varphi + rB_\varphi \nabla\varphi. \tag{21}$$

Since $\vec{B} \cdot \nabla\psi = 0$, ψ =const is a magnetic surface. We can now definitely orient the local magnetic coordinate system by setting

$$\vec{e}_n = \frac{\nabla\psi}{|\nabla\psi|}, \tag{22}$$

and adopt the space variables (ψ, χ, φ), which suit the axial symmetry of the magnetic surfaces. The variable χ is defined so that the local coordinate system

$$(\vec{e}_n, \vec{e}_p, \vec{e}_\varphi) \quad \text{with} \quad \vec{e}_p = \frac{\nabla\chi}{|\nabla\chi|} \tag{23}$$

is orthonormal. The transformation to the local magnetic coordinate system is given by

$$\vec{e}_{\parallel} = b_p \vec{e}_p + b_\varphi \vec{e}_\varphi \quad \text{where} \quad b_p^2 + b_\varphi^2 = 1. \tag{24}$$

Using Ampere's law and realizing that the current density up to first order in ϵ_e has no component along $\vec{e}_n$ shows that rB_φ is a surface quantity, that is

$$rB_\varphi = (rB_\varphi)(\psi). \tag{25}$$

Constraint (14) and the axial symmetry imply that F is also a surface quantity

$$F = F(\psi, \epsilon, \mu, \sigma). \tag{26}$$

By further using the axial symmetry and the above-mentioned properties of the current density, Eq.(19) yields

$$G^{(1)} = -\frac{rB_\varphi v_\parallel}{\Omega}\frac{\partial F_M}{\partial \psi}. \tag{27}$$

The stationary distribution function can finally be written in terms of the magnetic field

$$f_0 = F_M(1+\delta^{(1)}) + \frac{r}{\Omega}(v_b B_p - v_\parallel B_\varphi)\frac{\partial F_M}{\partial \psi} + \mathcal{O}(\epsilon_e^2). \tag{28}$$

The magnetic field however is partially created by the plasma itself. This self-consistency is expressed by Ampere's law

$$\nabla\times \vec{B} = \mu_0 \vec{j}_0 \quad \text{where} \quad \vec{j}_0(\vec{x}) = q\int \vec{v}\ f_0(\vec{x},\vec{v})d^3v. \tag{29}$$

To enable the calculation of the current density, the anisotropy $\delta^{(1)}$ must be specified. The charged particles follow the magnetic field lines, which twist round the magnetic surfaces. Because the kinetic energy and the magnetic moment are conserved quantities, some of the particles are trapped and oscillate on a portion of the magnetic field line. In a stationary state these particles cannot contribute to the current. The free particles however always follow the magnetic field line in the same direction and can thus generate a current. If $B_m(\psi)$ is the maximum magnetic field amplitude on the surface ψ =const, then a particle is

$$\begin{array}{lcl} \text{trapped} & \multirow{2}{*}{if} & \epsilon < B_m(\psi)\mu \\ \text{untrapped} & \text{if} & \epsilon > B_m(\psi)\mu. \end{array} \tag{30}$$

Consequently, we choose the following continuous definition for $\delta^{(1)}$

$$\delta^{(1)} = \frac{g\sigma}{v_{th}^2}(\epsilon - B_m\mu)\cdot H(\epsilon - B_m\mu), \tag{31}$$

where $g(\psi) \ll 1$ and H is the Heaviside function. For large values of ϵ this definition leads to negative values of the distribution function. The error however is negligible due to the fact that F_M is small in that region. The current density to first order in ϵ_e relative to Eq.(28) can now be evaluated. One obtains,

$$\vec{j}_0 = \frac{q}{2\sqrt{\pi}}\frac{gNv_{th}}{B_m}\ \vec{B} - r\frac{d}{d\psi}(NT)\ \vec{e}_\varphi + \mathcal{O}(\epsilon_e^2). \tag{32}$$

The first contribution is the force-free current along the magnetic field generated by the term $F_M\delta^{(1)}$. One can easily show that the continuity equation $\nabla \cdot \vec{j}_0 = 0$ is satisfied.

Inserting Eqs.(21) and (32) in Eq.(29) and projecting this equation along $\vec{e}_p$ and $\vec{e}_\varphi$ leads to the following equations

$$\begin{cases} -\Delta^*\psi \equiv -(\Delta - \frac{2}{r}\frac{\partial}{\partial r})\psi = \frac{1}{2}\frac{d}{d\psi}I^2 + \mu_0 r^2 \frac{d}{d\psi}p, \\ \frac{d}{d\psi}I = -\frac{\mu_0 q}{2\sqrt{\pi}}\frac{gNv_{th}}{B_m}, \end{cases} \tag{33}$$

where $I = rB_\varphi$ and p=NT is the scalar pressure. The first equation in Eqs.(33) is the well-known Grad-Shafranov equation of MHD theory.

III. Solution of the Linearized Vlasov Equation

By establishing a relation between the induced current and the electric field, the dielectric tensor describes the linear response of a system to a small electromagnetic perturbation. One must therefore solve the Vlasov equation linearized with respect to the perturbing quantities. Let $\tilde{f}$ and $(\vec{E}, \vec{\tilde{B}})$ be the fluctuating parts of the distribution function and electromagnetic fields respectively. The linearized Vlasov equation then reads

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} + \Omega(\vec{v} \times \vec{e}_{\|}) \cdot \frac{\partial}{\partial \vec{v}}\right) \tilde{f} = -\frac{q}{m}(\vec{E} + \vec{v} \times \vec{\tilde{B}}) \cdot \frac{\partial}{\partial \vec{v}} f_0 \tag{34}$$

f_0 being given by Eq.(28). The initial system being stationary and homogeneous in the toroidal direction, it is convenient to perform a Fourier transformation with respect to time and a Fourier series decomposition with respect to the variable φ. Let us choose the following definitions for these two transformations

$$g(t) = \frac{1}{2\pi}\int g(\omega)\exp(-i\omega t)d\omega, \tag{35}$$

$$g(\varphi) = \sum_{n=-\infty}^{+\infty} g_n \exp(in\varphi). \tag{36}$$

In what follows we will omit the index n, and define the toroidal wave number

$$k_\varphi = \frac{n}{r}. \tag{37}$$

After these transformations and adopting again the gyrokinetic variables, Eq.(34) can be written

$$\begin{aligned}&\left\{-i\omega + v_n\nabla_n + (v_\parallel b_p + v_b b_\varphi)\nabla_p + i(v_\parallel b_\varphi - v_b b_p)k_\varphi - \frac{1}{B}\left[v_\parallel \vec{v}\cdot(\nabla \vec{e}_\parallel)\cdot \vec{v} + \mu \vec{v}\cdot(\nabla B)\right]\frac{\partial}{\partial\mu}\right.\\ &\left.+\left[\vec{v}\cdot(\nabla \vec{e}_b)\cdot \vec{e}_n + \frac{v_\parallel}{v_\perp^2}\vec{v}\cdot(\nabla \vec{e}_\parallel)\cdot(\vec{v}_\perp\times\vec{e}_\parallel) - \Omega\right]\frac{\partial}{\partial\alpha}\right\}\tilde{f} = \\ &-\frac{q}{m}\left[(\vec{v}\cdot\vec{E})\frac{\partial}{\partial\epsilon} + \vec{v}_\perp\cdot(\vec{E} + \vec{v}\times\vec{\tilde{B}})\frac{1}{B}\frac{\partial}{\partial\mu} + \frac{1}{v_\perp^2}(\vec{e}_\parallel\times\vec{v}_\perp)\cdot(\vec{E} + \vec{v}\times\vec{\tilde{B}})\frac{\partial}{\partial\alpha}\right]f_0,\end{aligned} \tag{38}$$

where the electromagnetic fields obey Faraday's law

$$\nabla\times\vec{E} = -\frac{\partial\vec{\tilde{B}}}{\partial t}. \tag{39}$$

To respect causality, ω is assumed to have a small, positive, imaginary part.

Let us realize that it is not necessary to know the entire distribution function to evaluate the current density. To see this we decompose $\tilde{f}$ in a Fourier series with respect to α :

$$\tilde{f} = \sum_{l=-\infty}^{+\infty} \tilde{f}_l \exp(il\alpha). \tag{40}$$

The current density can then be written

$$\vec{j} = q\int d^3v\left[\frac{v_\perp}{2}(\tilde{f}_{-1} + \tilde{f}_1)\,\vec{e}_n + \frac{v_\perp}{2i}(\tilde{f}_{-1} - \tilde{f}_1)\,\vec{e}_b + v_\parallel\,\tilde{f}_0\vec{e}_\parallel\right], \tag{41}$$

where we use the following notation

$$\int d^3v = 2\pi\int_0^{+\infty} v_\perp dv_\perp \int_{-\infty}^{+\infty} dv_\parallel. \tag{42}$$

We thus solve Eq.(38) in such a way as to provide the three Fourier coefficients $\tilde{f}_0, \tilde{f}_{-1}$ and $\tilde{f}_1$.

Eq.(38) is solved up to the orders ϵ_p^2 and $\epsilon_e\epsilon_p$, where ϵ_p evaluates the variation of the electric field in the poloidal plane with respect to the Larmor radius ρ

$$\epsilon_p = \rho\frac{\nabla_\nu E}{E} \quad , \quad \nu\in\{n,p\}. \tag{43}$$

The terms of order ϵ_e^2 are neglected so as to stay consistent with the unperturbed state derived only up to first order in ϵ_e. The wave number k_φ is of order ϵ_e

$$k_\varphi = \mathcal{O}(\epsilon_e). \tag{44}$$

The poloidal component of the magnetostatic field is assumed to be small, therefore we set

$$b_p = \mathcal{O}(\epsilon_e). \tag{45}$$

As with the stationary state, we expand $\tilde{f}$ in a series with respect to ϵ (ϵ_e or ϵ_p)

$$\tilde{f} = \tilde{f}^{(0)} + \tilde{f}^{(1)} + \dots \tag{46}$$

To be able to write Eq.(38) to a certain order in ϵ we must evaluate the leading orders of the geometric coefficients $\beta_{\mu\nu}$. This is done in the appendix. We obtain the following relations :

$$\begin{aligned} \beta_{nn} = 0, \quad \beta_{n||}, \beta_{||n}, \beta_{||b} = \mathcal{O}(\epsilon_e) \quad \beta_{b||} = \mathcal{O}(\epsilon_p), \\ \beta_{nb}, \beta_{bn}, \beta_{bb} = \mathcal{O}(\epsilon_e^2), \quad \beta_{||||} = \mathcal{O}(\epsilon_e \epsilon_p). \end{aligned} \tag{47}$$

For similar reasons we must consider the following derivatives of the coefficients

$$\begin{aligned} &\nabla_p \beta_{||n} = \beta_{||b}\beta_{b||} = \mathcal{O}(\epsilon_e\epsilon_p), \quad \nabla_p \beta_{||b} = -\beta_{||n}\beta_{b||} = \mathcal{O}(\epsilon_e\epsilon_p), \\ &\nabla_n \beta_{b||}, \nabla_p \beta_{b||} = \mathcal{O}(\epsilon_e\epsilon_p). \end{aligned} \tag{48}$$

Subsequently we also use the relations

$$\nabla_n \ln k_\varphi = \nabla_n \ln \Omega = \beta_{||b}, \qquad \nabla_p \ln k_\varphi = \nabla_p \ln \Omega = -\beta_{||n}, \tag{49}$$

which are valid to the order we consider, obtained using Eq.(A.1) and the fact that to order zero in ϵ_e the magnetostatic field decreases like $1/r$.

Let us write $\tilde{f}_l^{(m)}$ as the contribution of order m in ϵ to the harmonic l of the Fourier series. To solve Eq.(38), we first write this equation to the different orders in ϵ. We then transform the resulting equations using the Fourier series decomposition with respect to α, which enables us to express the contributions $\tilde{f}_l^{(m)}$ in terms of the contributions of lower order with respect to ϵ.

To zeroth order in ϵ Eq.(38) reduces to

$$\tilde{\mathcal{L}}^{(0)} \tilde{f}^{(0)} = -\frac{q}{m}(\vec{v} \cdot \vec{E}) \frac{\partial}{\partial \epsilon} F, \tag{50}$$

where

$$\tilde{\mathcal{L}}^{(0)} = -i(\omega - k_\varphi v_{||} - i\Omega \frac{\partial}{\partial \alpha}). \tag{51}$$

In Eq.(51) k_φ is of order zero, despite the ordering (44), as $k_\varphi v_\|$ can be of the same order as ω. After applying the Fourier series transformation, Eq.(50) yields

$$\begin{aligned}\overset{\sim(0)}{f_0} &= \frac{q}{im\Omega_0} v_\| \frac{\partial F}{\partial \epsilon} E_\|,\\ \overset{\sim(0)}{f_1} &= \frac{q}{im\Omega_{-1}} \frac{v_\perp}{2} \frac{\partial F}{\partial \epsilon}(E_n - iE_b),\end{aligned} \tag{52}$$

where we have used the definition

$$\Omega_l = \omega - k_\varphi v_\| - l\Omega. \tag{53}$$

In Eqs.(52) F is given by Eqs.(16) and (31); the contributions $\overset{\sim(0)}{f}$ thus already contain terms of first order. This is done for practical reasons so that we do not have to decompose F in terms of various orders. The same is done with the coefficients $\beta_{\mu\nu}$, which are classified to their leading orders. Also, we do not write the expressions for $\overset{\sim(m)}{f_{-l}}$ because they can be obtained from the reality condition

$$\overset{\sim(m)}{f_l}(\omega, k_\varphi) = \overset{\sim(m)*}{f_{-l}}(-\omega, -k_\varphi) \quad , \quad \omega \in \Re. \tag{54}$$

Now to first order in ϵ Eq.(38) reads

$$\begin{aligned}&\overset{\sim(0)}{\mathcal{L}}\overset{\sim(1)}{f} + \overset{\sim(1)}{\mathcal{L}}\overset{\sim(0)}{f} =\\ &-\frac{q}{m}\Big[(\vec{v}\cdot\vec{E})\frac{\partial}{\partial\epsilon}F^{(1)} + \vec{v}_\perp\cdot(\vec{E} + \vec{v}\times\vec{\tilde{B}}^{(0)})\frac{1}{B}\frac{\partial}{\partial\mu}(F + F^{(1)})\\ &+\frac{1}{v_\perp^2}(\vec{e}_\| \times \vec{v}_\perp)\cdot(\vec{E} + \vec{v}\times\vec{\tilde{B}}^{(0)})\frac{\partial}{\partial\alpha}F^{(1)}\Big],\end{aligned} \tag{55}$$

with

$$\overset{\sim(1)}{\mathcal{L}} = \Big\{v_n\nabla_n + v_b\nabla_p - \frac{1}{B}\left[v_\|^2(v_n\beta_{\|b} - v_b\beta_{\|n}) + \mu\,\vec{v}\cdot(\nabla B)\right]\frac{\partial}{\partial\mu}. \tag{56}$$

The components of the magnetic field to zeroth order in ϵ expressed in terms of the electric field using Eq.(39) are given by

$$\overset{\sim(0)}{B_n} = -\frac{k_\varphi}{\omega}E_b, \quad \overset{\sim(0)}{B_b} = \frac{k_\varphi}{\omega}E_n, \quad \overset{\sim(0)}{B_\|} = 0, \tag{57}$$

where as in Eq.(51) k_φ is of order zero. Solving Eq.(55) using the zero order contributions and Eq.(28) yields

$$
\begin{aligned}
\overset{\sim(1)}{f_0} &= -\frac{q}{4m\Omega_0}\Big\{v_\perp^2\left[Q_1^+(\nabla_n E_n + \nabla_p E_b) + iQ_1^-(\nabla_n E_b - \nabla_p E_n)\right] \\
&+ (v_\perp^2\beta_{b\|} - 2v_\|^2\beta_{\|b})(Q_1^+ E_n + iQ_1^- E_b) \\
&+ (v_\perp^2\beta_{n\|} - 2v_\|^2\beta_{\|n})(iQ_1^- E_n - Q_1^+ E_b) \\
&+ v_\perp^2(2k_\varphi v_\|\beta_{\|b} + i\Omega\beta_{\|n})(Q_2^+ E_n + Q_2^- E_b) \\
&+ v_\perp^2(2k_\varphi v_\|\beta_{\|n} - i\Omega\beta_{\|b})(Q_2^- E_n - Q_2^+ E_b)\Big\}\frac{\partial F}{\partial \epsilon} \\
&-i\frac{qE_b}{m\Omega\Omega_0}\left[\frac{v_\perp^2}{2}\nabla_n\frac{\partial F}{\partial\epsilon} + \frac{\Omega_0}{\omega}\nabla_n F\right] + i\frac{qE_\|}{m\Omega\Omega_0 b_p}\left[v_\|^2\nabla_n\frac{\partial F}{\partial\epsilon} + \nabla_n F\right],
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
\overset{\sim(1)}{f_1} &= -\frac{qv_\perp v_\|}{2m\Omega_0\Omega_{-1}}\left[\frac{k_\varphi v_\| + \omega}{\Omega_0}R^+ + \mathcal{L}^-\right]E_\|\frac{\partial F}{\partial\epsilon} \\
&+\frac{iqv_\perp}{2m\Omega_{-1}}\left[-\frac{1}{B}\frac{\Omega_0}{\omega}\frac{\partial F}{\partial\mu} + \frac{1}{\omega\Omega}\frac{k_\varphi}{b_p}(\nabla_n F) + \frac{v_\|}{\Omega b_p}\left(\nabla_n\frac{\partial F}{\partial\epsilon}\right)\right](E_n - iE_b) \\
&-\frac{qv_\perp v_\|}{2m\Omega\Omega_{-1}}E_\|\nabla_n\frac{\partial F}{\partial\epsilon},
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
\overset{\sim(1)}{f_2} &= -\frac{qv_\perp^2}{4m\Omega_{-1}\Omega_{-2}}\left[-(i\beta_{n\|} + \beta_{b\|}) + \frac{2k_\varphi v_\| - \Omega}{\Omega_{-1}}R^+ + \mathcal{L}^-\right](E_n - iE_b)\frac{\partial F}{\partial\epsilon} \\
&-\frac{qv_\perp^2}{4m\Omega\Omega_{-2}}(En - iE_b)\nabla_n\frac{\partial F}{\partial\epsilon},
\end{aligned}
\tag{60}
$$

where we have used the notations

$$
\begin{aligned}
&Q_1^\pm = \left(\frac{1}{\Omega_1} \pm \frac{1}{\Omega_{-1}}\right), \quad Q_2^\pm = \left(\frac{1}{{\Omega_1}^2} \pm \frac{1}{{\Omega_{-1}}^2}\right), \\
&R^\pm = \beta_{\|b} \pm i\beta_{\|n}, \qquad \mathcal{L}^\pm = \nabla_n \pm i\nabla_p.
\end{aligned}
\tag{61}
$$

Although the coefficients $\overset{\sim(1)}{f_{-2}}$ and $\overset{\sim(1)}{f_2}$ are not needed for calculating the current density they must be established for evaluating the contributions of second order in ϵ.

Finally to second order in ϵ Eq.(38) can be written

$$
\begin{aligned}
&\overset{\sim(0)}{\mathcal{L}}\overset{\sim(2)}{f} + \overset{\sim(1)}{\mathcal{L}}\overset{\sim(1)}{f} + \overset{\sim(2)}{\mathcal{L}}\overset{\sim(0)}{f} = \\
&\quad -\frac{q}{m}\left[\vec{v}_\perp\cdot(\vec{v}\times\vec{\tilde{B}}^{(1)})\frac{1}{B}\frac{\partial}{\partial\mu}(F + F^{(1)}) + \frac{1}{v_\perp^2}(\vec{e}_\| \times \vec{v}_\perp)\cdot(\vec{v}\times\vec{\tilde{B}}^{(1)})\frac{\partial}{\partial\alpha}F^{(1)}\right],
\end{aligned}
\tag{62}
$$

where

$$\overset{\sim(2)}{\mathcal{L}} = v_{\|}(b_p\nabla_p - \beta_{\|\|\|}\frac{\partial}{\partial\alpha}), \tag{63}$$

and

$$\overset{\sim(1)}{B_n} = \frac{1}{i\omega}\nabla_p E_{\|}, \quad \overset{\sim(1)}{B_b} = -\frac{1}{i\omega}\nabla_n E_{\|}, \quad \overset{\sim(1)}{B_{\|}} = \frac{1}{i\omega}(\nabla_n E_b - \nabla_p E_n + \beta_{b\|}E_b). \tag{64}$$

Solving Eq.(62) using the lower order contributions and Eq.(28)leads to

$$\begin{aligned}
\frac{m}{q}\overset{\sim(2)}{f_0} = & \left\{ \frac{v_\perp^2 v_\|}{4\Omega_0^2}\left[iQ_1^+(\nabla_n\nabla_n + \nabla_p\nabla_p) - Q_1^-(\nabla_n\nabla_p - \nabla_p\nabla_n)\right]\frac{\partial F}{\partial\epsilon} \right.\\
& + \frac{k_\varphi v_\perp^2 v_\|^2}{2\Omega_0^2}\frac{\partial F}{\partial\epsilon}\left[iQ_2^+\mathcal{O}^- - Q_2^-\mathcal{O}^+\right] + \frac{\Omega v_\perp^2 v_\|}{4\Omega_0^2}\frac{\partial F}{\partial\epsilon}\left[iQ_2^-\mathcal{O}^- - Q_2^+\mathcal{O}^+\right]\\
& + \frac{v_\perp^2 v_\|}{4\Omega_0^2}\beta_{b\|}(iQ_1^+\nabla_n - Q_1^-\nabla_p)\frac{\partial F}{\partial\epsilon} + i(k_\varphi v_\| + \omega)\frac{v_\perp^2 v_\|}{2\Omega_0^3}\frac{\partial F}{\partial\epsilon}Q_1^+\mathcal{O}^-\\
& + (2v_\|^2\beta_{\|n} - v_\perp^2\beta_{n\|})\frac{v_\|}{4\Omega_0^2}\frac{\partial F}{\partial\epsilon}(iQ_1^+\nabla_p + Q_1^-\nabla_n)\\
& - \frac{v_\perp^2 v_\|}{4\Omega\Omega_0}\left(\nabla_n\frac{\partial F}{\partial\epsilon}\right)\left(Q_1^+\nabla_p + iQ_1^-(\nabla_n + \beta_{b\|})\right)\\
& \left. - \frac{v_\|^3}{2\Omega_0^2}\beta_{\|b}\frac{\partial F}{\partial\epsilon}(iQ_1^+\nabla_n - Q_1^-\nabla_p) - \left(b_p\frac{v_\|^2}{\Omega_0^2}\frac{\partial F}{\partial\epsilon} + \frac{v_\|}{\omega\Omega\Omega_0}(\nabla_n F)\right)\nabla_p\right\} E_\|\\
& - i\frac{v_\perp^2}{4\Omega_0}\left(\frac{v_\|}{\Omega b_p}\left(\nabla_n\frac{\partial F}{\partial\epsilon}\right) - \frac{\Omega_0}{\omega B}\frac{\partial F}{\partial\mu} + \frac{k_\varphi}{\omega\Omega b_p}(\nabla_n F)\right) \times (iQ_1^+T_1^+ - Q_1^-T_1^-),
\end{aligned} \tag{65}$$

$$\begin{aligned}
i\frac{m}{q}\overset{\sim(2)}{f_1} = & \left\{\left[\frac{-v_\perp^3}{8\Omega_{-1}^2\Omega_{-2}}\left\{\left[\mathcal{L}^+ + 2\frac{k_\varphi v_\| - \Omega}{\Omega_{-2}}R^- + 2(\beta_{b\|} - i\beta_{n\|})\right](\mathcal{L}^- - \beta_{b\|})\right.\right.\right.\\
& \left. + \frac{2k_\varphi v_\| - \Omega}{\Omega_{-1}}\left(R^+(\mathcal{L}^+ + \beta_{b\|}) + R^-(\mathcal{L}^- - \beta_{b\|})\right) - i\beta_{n\|}(\mathcal{L}^+ + 2\beta_{b\|})\right\}\\
& \left. + \frac{v_\perp v_\|^2}{2\Omega_{-1}^2\Omega_{-2}}R^-(\mathcal{L}^- - \beta_{b\|})\right] \times (E_n - iE_b)\\
& + \frac{v_\perp^3}{8\Omega_0\Omega_{-1}}\left\{-(Q_1^+\mathcal{L}^-T_1^+ + iQ_1^-\mathcal{L}^-T_1^-)\right.\\
& - \frac{2k_\varphi v_\|}{\Omega_0}R^+(Q_1^+T_1^+ + iQ_1^-T_1^-) + \beta_{n\|}(Q_1^+\mathcal{L}^-E_b - iQ_1^-\mathcal{L}^-E_n)\\
& + (2k_\varphi v_\| Q_2^+ + \Omega Q_2^-)(\beta_{\|n}\mathcal{L}^-E_b - \beta_{\|b}\mathcal{L}^-E_n - R^+T_1^+ - i\beta_{b\|}T_2^+)\\
& \left. - i(2k_\varphi v_\| Q_2^- + \Omega Q_2^+)(\beta_{\|n}\mathcal{L}^-E_n + \beta_{\|b}\mathcal{L}^-E_b + R^+T_1^- - i\beta_{b\|}T_2^-)\right\}\\
& \left. + \frac{v_\perp v_\|^2}{4\Omega_0\Omega_{-1}}\left[Q_1^+\left(\mathcal{L}^-T_2^- + R^+T_1^+\right) + iQ_1^-\left(\mathcal{L}^-T_2^+ + R^+T_1^-\right)\right] \right\}\frac{\partial F}{\partial\epsilon}
\end{aligned} \tag{66}$$

$$-\frac{v_\perp^3}{8\Omega\Omega_{-1}\Omega_{-2}}\left(\nabla_n\frac{\partial F}{\partial\epsilon}\right)(\mathcal{L}^+ + 2\beta_{b\|})(E_n - iE_b) + \frac{iv_\perp v_\|}{2\Omega_{-1}^2}\frac{\partial F}{\partial\epsilon}(i\beta_{\|\|\|} - b_p\nabla_p)(E_n - iE_b)$$
$$+i\mathcal{L}^-\frac{v_\perp}{2\Omega\Omega_0\Omega_{-1}}\left[E_\|\frac{1}{b_p}\left(v_\|^2\nabla_n\frac{\partial F}{\partial\epsilon} + \nabla_n F\right) - E_b\left(\frac{v_\perp^2}{2}\nabla_n\frac{\partial F}{\partial\epsilon} + \frac{\Omega_0}{\omega}\nabla_n F\right)\right]$$
$$+i\frac{v_\perp}{2\omega\Omega_{-1}}\left[\frac{(\nabla_n F)}{\Omega}T_1^- - \left(\frac{1}{\Omega b_p}(\nabla_n F) + \frac{v_\|}{B}\frac{\partial F}{\partial\mu}\right)\mathcal{L}^- E_\|\right],$$

with the notations

$$\begin{aligned}
T_1^+ &= \left((\nabla_n + \beta_{b\|})E_n + \nabla_p E_b\right), & T_1^- &= \left((\nabla_n + \beta_{b\|})E_b - \nabla_p E_n\right), \\
T_2^+ &= \beta_{\|n}E_n + \beta_{\|b}E_b, & T_2^- &= \beta_{\|b}E_n - \beta_{\|n}E_b, \\
\mathcal{O}^+ &= \beta_{\|n}\nabla_n + \beta_{\|b}\nabla_p, & \mathcal{O}^- &= \beta_{\|b}\nabla_n - \beta_{\|n}\nabla_p.
\end{aligned} \tag{67}$$

IV. Dielectric Tensor Operator

To obtain the dielectic tensor operator we still have to integrate $\tilde{f}$ over velocity space to evaluate the induced current density (41). Then, using the definition of the conductivity tensor $\overleftrightarrow{\sigma}$

$$\vec{j} = \overleftrightarrow{\sigma}\vec{E}, \tag{68}$$

we finally get the dielectric tensor

$$\overleftrightarrow{\epsilon} = \overleftrightarrow{1} - \frac{1}{i\omega\epsilon_0}\overleftrightarrow{\sigma}, \tag{69}$$

where $\overleftrightarrow{1}$ is the unit tensor. The dielectric tensor is established in terms of its components relative to the local magnetic coordinate system, a natural choice with respect to the physical properties of the plasma. The integration is performed in the cylindrical variables $(v_\perp, v_\|)$. The integrations over $v_\perp$ are performed first and are quite straightforward. The integrations relative to $v_\|$ are somewhat more lengthy. By repeated use of the relations

$$\frac{1}{\Omega_l\Omega_{l'}} = \frac{1}{(l-l')\Omega}\left(\frac{1}{\Omega_l} - \frac{1}{\Omega_{l'}}\right) \qquad l \neq l' \tag{70}$$

$$\frac{1}{\Omega_l^2} = \frac{1}{l}\frac{\partial}{\partial\Omega}\frac{1}{\Omega_l} \qquad l \neq 0, \qquad \frac{v_\|}{\Omega_l^2} = \frac{\partial}{\partial k_\varphi}\frac{1}{\Omega_l},$$

we can reduce all these integrals to the two dispersion functions :

- Maxwellian distribution (Shafranov[5] definition) :

$$Z^{sh}(z) = \frac{z}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{1}{z-x} e^{-x^2} dx \qquad \text{Im } z > 0. \tag{71}$$

- Force-free current :

$$Y(z) = \int_{-\infty}^{+\infty} \frac{sgn(x)}{z-x} e^{-x^2} dx \qquad \text{Im } z > 0 \tag{72}$$

For convenience we define

$$Z_l^{sh} = Z^{sh}(z_l) \quad \text{with} \quad z_l = \frac{\omega - l\Omega}{|k_\varphi| v_{th}}, \tag{73}$$

$$\tilde{Z}_l = \frac{\omega_p^2}{\omega - l\Omega} Z_l^{sh}, \qquad Z_l = \frac{T}{m} \tilde{Z}_l, \tag{74}$$

$$Y_l = Y(z_l), \tag{75}$$

$$Y_l^\lambda = Y(\lambda^{\frac{1}{2}} \gamma z_l) \quad \text{with} \quad \lambda = \frac{B_m}{B} \quad \text{and} \quad \gamma^2 = (\lambda - 1)^{-1}, \tag{76}$$

where ω_p is the local plasma frequency. In relations containing the term $\nabla_n F_M E_\nu$ we can replace ∇_n by the new operator ∇'_n, which by definition operates on ψ and $\vec{E}$ but not on B and k_φ. This convention allows us to commute the operator with the integral sign. Putting the results obtained in this way in the most elegant form using the recurrence equation

$$\frac{d}{dz} Z^{sh}(z) = \frac{1}{z} Z^{sh}(z) - 2z(Z^{sh}(z) - 1), \tag{77}$$

and the relation

$$\beta_{p\varphi} \nabla_p + \beta_{n\varphi} \nabla_n + \nabla_n \nabla_p - \nabla_p \nabla_n = 0, \tag{78}$$

the dielectric tensor can finally be written as follows.

- Dielectric tensor in zero order in ϵ :

$$\epsilon_{nn}^{(0)} = \epsilon_{bb}^{(0)} = 1 - \frac{1}{2\omega}(\tilde{Z}_1 + \tilde{Z}_{-1}), \tag{79}$$

$$\epsilon_{\|\|}^{(0)} = 1 + \frac{2}{(k_\varphi v_{th})^2} \left[\omega_p^2 - \omega \, \tilde{Z}_0 \right], \tag{80}$$

$$\epsilon_{nb}^{(0)} = -\epsilon_{bn}^{(0)} = -\frac{i}{2\omega}(\tilde{Z}_1 - \tilde{Z}_{-1}), \tag{81}$$

$$\epsilon_{n\|}^{(0)} = \epsilon_{\|n}^{(0)} = \epsilon_{b\|}^{(0)} = \epsilon_{\|b}^{(0)} = 0. \tag{82}$$

- Contribution to the dielectric tensor of first order in ϵ :

$$\begin{aligned}
\epsilon_{nn}^{(1)} &= \epsilon_{bb}^{(1)} = \frac{k_\varphi}{2\omega\Omega b_p}\left\{\nabla'_n\left[\frac{\partial}{\partial\Omega}(Z_1 - Z_{-1}) - \frac{1}{\omega}(Z_1 + Z_{-1})\right]\right\} \\
&\qquad + \frac{K}{2}\left[X_1 + X_{-1} + 4 + \lambda\frac{\Omega}{\omega}(W_1 - W_{-1})\right], \qquad (83)\\
\epsilon_{nb}^{(1)} &= -\epsilon_{bn}^{(1)} = \frac{i}{b_p}\left\{\nabla'_n\left[B_{n\|} + \frac{k_\varphi}{2\omega^2\Omega}(Z_{-1} - Z_1)\right]\right\} \\
&\qquad + i\frac{K}{2}\left[X_1 - X_{-1} + \lambda\frac{\Omega}{\omega}(W_1 + W_{-1} - \frac{2}{\lambda})\right], \qquad (84)\\
\epsilon_{\|\|}^{(1)} &= \frac{1}{\Omega b_p}\left(\nabla'_n\frac{\partial\tilde{Z}_0}{\partial k_\varphi}\right) + 2K\left(\frac{\omega}{k_\varphi v_{th}}\right)^2\left[\lambda(Y_0 - Y_0^\lambda) - \left(\frac{\omega}{k_\varphi v_{th}}\right)^2 Y_0 + 1\right], \qquad (85)\\
\epsilon_{n\|}^{(1)} &= B_{b\|}\left(\nabla_p + \beta_{\|n}\right) + iB_{n\|}\left(\nabla_n - \beta_{\|b}\right) \\
&\qquad + \frac{2}{\Omega}\left[\omega\left(\beta_{\|n}B_{n\|} - i\beta_{\|b}B_{b\|}\right) + \frac{1}{k_\varphi}\left(\nabla_p\tilde{Z}_0\right)\right], \qquad (86)\\
\epsilon_{\|n}^{(1)} &= \left(-\nabla_p + \beta_{n\|}\right)B_{b\|} + i\left(\nabla_n + \beta_{b\|}\right)B_{n\|} \\
&\qquad + \frac{2}{\Omega}\left[\omega\left(\beta_{\|n}B_{n\|} + i\beta_{\|b}B_{b\|}\right) + \frac{1}{k_\varphi}\left(\nabla_p\tilde{Z}_0\right)\right], \qquad (87)\\
\epsilon_{b\|}^{(1)} &= B_{b\|}\left(-\nabla_n + \beta_{\|b}\right) + iB_{n\|}\left(\nabla_p + \beta_{\|n}\right) \\
&\qquad + \frac{2\omega}{\Omega}\left(\beta_{\|b}B_{n\|} + i\beta_{\|n}B_{b\|}\right) + G_{b\|}, \qquad (88)\\
\epsilon_{\|b}^{(1)} &= \left(\nabla_n + \beta_{b\|}\right)B_{b\|} + i\left(\nabla_p - \beta_{n\|}\right)B_{n\|} \\
&\qquad + \frac{2\omega}{\Omega}\left(\beta_{\|b}B_{n\|} - i\beta_{\|n}B_{b\|}\right) + G_{b\|}. \qquad (89)
\end{aligned}$$

- Contribution to the dielectric tensor of order $\epsilon_e\epsilon_p$ and ϵ_p^2 :

$$\overleftrightarrow{\epsilon}^{(2)} = \overleftrightarrow{\epsilon}^{R} + i\,\overleftrightarrow{\epsilon}^{I}, \qquad (90)$$

$$\begin{aligned}
\epsilon_{nn}^{R} &= \left(\nabla_n + \beta_{b\|} - \beta_{\|b}\right)A_{nn}\nabla_n + \left(\nabla_p - \beta_{n\|} + \beta_{\|n}\right)A_{bb}\nabla_p \\
&\quad \left(\nabla_n\beta_{b\|}\right)C_1^{nn} + \beta_{b\|}^2 C_2^{nn} + \beta_{\|b}\beta_{b\|}C_3^{nn}, \qquad (91)\\
\epsilon_{nn}^{I} &- \epsilon_{bb}^{I} = 2D_2^{nn}\left[\beta_{\|b}\nabla_p - \beta_{\|n}\left(\nabla_n + \beta_{b\|}\right)\right] + \\
&\quad \frac{1}{\omega\Omega^2}\left[\nabla'_n(Z_{-1} - Z_1)\right]\nabla_p, \qquad (92)
\end{aligned}$$

$$\begin{aligned}
\epsilon^R_{bb} &= \left(\nabla_n + \beta_{b\|} - \beta_{\|b}\right) A_{bb}\nabla_n + \left(\nabla_p - \beta_{n\|} + \beta_{\|n}\right) A_{nn}\nabla_p \\
&\quad \left(\nabla_n\beta_{b\|}\right) C_1^{bb} + \beta_{b\|}^2 C_2^{nn} + \beta_{\|b}\beta_{b\|}C_3^{bb} + \frac{\beta_{\|\|}}{2\omega}\frac{\partial}{\partial k_\varphi}(\tilde{Z}_{-1} - \tilde{Z}_1), \qquad (93)\\
\epsilon^I_{bb} &= A_{nb}\left[2\beta_{n\|}\nabla_n + B_{b\|}\left(2\nabla_p + \beta_{\|n} + (\nabla_p \ln A_{nb})\right) + \left(\nabla_p\beta_{b\|}\right)\right] + \\
&\quad D_1^{nn}\left(\beta_{\|b}\nabla_p + \beta_{\|n}\nabla_n\right) - D_2^{nn}\left[\beta_{\|b}\nabla_p - \beta_{\|n}\left(\nabla_n + \beta_{b\|}\right)\right] + D_3^{bb}\nabla_p, \qquad (94)
\end{aligned}$$

$$\begin{aligned}
\epsilon^R_{\|\|} &= \left(\nabla_n + \beta_{b\|} - \beta_{\|b}\right) A_{\|\|}\nabla_n + \left(\nabla_p - \beta_{n\|} + \beta_{\|n}\right) A_{\|\|}\nabla_p, \qquad (95)\\
\epsilon^I_{\|\|} &= D_1^{\|\|}\left(\beta_{\|b}\nabla_p + \beta_{\|n}\nabla_n\right) + D_2^{\|\|}\nabla_p, \qquad (96)
\end{aligned}$$

$$\begin{aligned}
\epsilon^R_{nb} &= -A_{nn}\left[2\beta_{n\|}\nabla_n + B_{b\|}\left(2\nabla_p + \beta_{\|n} + (\nabla_p \ln A_{nn})\right) + \left(\nabla_p\beta_{b\|}\right)\right] \\
&\quad + B_{nb}\nabla_p\left(\nabla_n + \beta_{b\|}\right) + C_1^{nb}\left(\beta_{\|b}\nabla_p + \beta_{\|n}\nabla_n\right) \\
&\quad + C_2^{nb}\left[\beta_{\|b}\nabla_p - \beta_{\|n}\left(\nabla_n + \beta_{b\|}\right)\right] + C_3^{nb}\nabla_p - C_2^{nb}\beta_{b\|}\beta_{\|n}, \qquad (97)\\
\epsilon^I_{nb} &= \left(\nabla_n + \beta_{b\|} - \beta_{\|b}\right) A_{nb}\nabla_n + \left(\nabla_p - \beta_{n\|} + \beta_{\|n}\right) A_{nb}\nabla_p \\
&\quad + \left(\nabla_n\beta_{b\|}\right) D_1^{nb} + \beta_{b\|}^2 D_2^{nb} - \beta_{\|b}\beta_{b\|}D_1^{nn} - D_2^{nn}\left(\beta_{\|b}\nabla_n + \beta_{\|n}\nabla_p + \beta_{\|b}\beta_{b\|}\right) \\
&\quad + \frac{1}{2\omega\Omega^2}\left[\nabla'_n(Z_1 - Z_{-1})\right]\nabla_n - \frac{\beta_{\|\|}}{2\omega}\frac{\partial}{\partial k_\varphi}(\tilde{Z}_1 + \tilde{Z}_{-1}), \qquad (98)
\end{aligned}$$

$$\begin{aligned}
\epsilon^R_{bn} &= A_{nn}\left[2\beta_{n\|}\nabla_n + B_{b\|}\left(2\nabla_p + \beta_{\|n} + (\nabla_p \ln A_{nn})\right) + \left(\nabla_p\beta_{b\|}\right)\right] \\
&\quad + \left(\nabla_n - \beta_{\|b}\right)\nabla_p B_{nb} + (\beta_{\|n} - \beta_{n\|})B_{nb}\nabla_n - C_1^{nb}\left(\beta_{\|b}\nabla_p + \beta_{\|n}\nabla_n\right) \\
&\quad - C_2^{nb}\left[\beta_{\|b}\nabla_p - \beta_{\|n}\left(\nabla_n + \beta_{b\|}\right)\right] - C_3^{nb}\nabla_p - C_2^{nb}\beta_{b\|}\beta_{\|n}, \qquad (99)\\
\epsilon^I_{bn} &= -\left(\nabla_n + \beta_{b\|} - \beta_{\|b}\right) A_{nb}\nabla_n - \left(\nabla_p - \beta_{n\|} + \beta_{\|n}\right) A_{nb}\nabla_p \\
&\quad - \left(\nabla_n\beta_{b\|}\right) D_1^{nb} - \beta_{b\|}^2 D_2^{nb} + \beta_{\|b}\beta_{b\|}D_1^{nn} - D_2^{nn}\left(\beta_{\|b}\nabla_n + \beta_{\|n}\nabla_p + \beta_{\|b}\beta_{b\|}\right) \\
&\quad + \frac{1}{2\omega\Omega^2}\left[\nabla'_n(Z_1 - Z_{-1})\right]\nabla_n + \frac{\beta_{\|\|}}{2\omega}\frac{\partial}{\partial k_\varphi}(\tilde{Z}_1 + \tilde{Z}_{-1}), \qquad (100)
\end{aligned}$$

$$\begin{aligned}
\epsilon^R_{n\|} &= \left(C_{n\|} - \frac{KL_2}{2\Omega k_\varphi}\right)\nabla_p, \qquad (101)\\
\epsilon^I_{n\|} &= \left(D_{n\|} + \frac{KL_1}{2\Omega k_\varphi}\right)\nabla_n, \qquad (102)
\end{aligned}$$

$$\epsilon^R_{\parallel n} = -\epsilon^R_{n\parallel}, \quad (103)$$
$$\epsilon^I_{\parallel n} = D_{n\parallel}\nabla_n + \frac{KL_1}{2\Omega k_\varphi}\left(\nabla_n + \beta_{b\parallel}\right), \quad (104)$$

$$\epsilon^R_{b\parallel} = \left(-C_{n\parallel} + \frac{KL_2}{2\Omega k_\varphi}\right)\nabla_n, \quad (105)$$
$$\epsilon^I_{b\parallel} = \left(D_{n\parallel} + \frac{KL_1}{2\Omega k_\varphi}\right)\nabla_p, \quad (106)$$

$$\epsilon^R_{\parallel b} = C_{n\parallel}\nabla_n - \frac{KL_2}{2\Omega k_\varphi}\left(\nabla_n + \beta_{b\parallel}\right), \quad (107)$$
$$\epsilon^I_{\parallel b} = \epsilon^I_{b\parallel}. \quad (108)$$

We have used the following notations :

- Terms relative to the Maxwellian distribution :

$$A_{nn} = \frac{1}{2\omega\Omega^2}(Z_2 + Z_{-2} - Z_1 - Z_{-1}), \quad (109)$$
$$A_{bb} = \frac{1}{2\omega\Omega^2}(Z_2 + Z_{-2} - 3Z_1 - 3Z_{-1} + 4Z_0), \quad (110)$$
$$A_{\parallel\parallel} = \frac{1}{2\omega\Omega^2 k_\varphi^2}\left[(\omega-\Omega)^2\,\tilde{Z}_1 + (\omega+\Omega)^2\,\tilde{Z}_{-1} - 2\omega^2\,\tilde{Z}_0\right], \quad (111)$$
$$A_{nb} = \frac{1}{2\omega\Omega^2}(Z_2 - Z_{-2} - 2Z_1 + 2Z_{-1}), \quad (112)$$
$$B_{n\parallel} = \frac{1}{2\omega\Omega k_\varphi}\left[(\omega-\Omega)\,\tilde{Z}_1 - (\omega+\Omega)\,\tilde{Z}_{-1}\right], \quad (113)$$
$$B_{b\parallel} = \frac{1}{2\omega\Omega k_\varphi}\left[2\omega\,\tilde{Z}_0 - (\omega-\Omega)\,\tilde{Z}_1 - (\omega+\Omega)\,\tilde{Z}_{-1}\right], \quad (114)$$
$$B_{nb} = \frac{1}{\omega\Omega^2}(Z_1 + Z_{-1} - 2Z_0), \quad (115)$$
$$C_1^{nn} = \frac{1}{\omega\Omega}\frac{\partial}{\partial\Omega}(Z_1 + Z_{-1}) - A_{nn}, \quad (116)$$
$$C_2^{nn} = C_1^{nn} - A_{nn}, \quad (117)$$
$$C_3^{nn} = \frac{3}{2\omega\Omega^2}(2Z_0 - Z_2 - Z_{-2}) + \frac{1}{2\omega\Omega^2}k_\varphi\frac{\partial}{\partial k_\varphi}(2Z_0 - Z_2 - Z_{-2})$$
$$-\frac{1}{2\omega\Omega}\frac{\partial}{\partial\Omega}(Z_2 + Z_{-2}) + \frac{4\omega}{\Omega}A_{nb} - \frac{1}{k_\varphi}\frac{\partial}{\partial k_\varphi}k_\varphi B_{n\parallel}, \quad (118)$$
$$C_1^{bb} = \frac{1}{2\omega\Omega}\left[\frac{1}{\Omega}(4Z_0 - Z_1 - Z_{-1} - Z_2 - Z_{-2}) + 2\frac{\partial}{\partial\Omega}(Z_1 + Z_{-1})\right], \quad (119)$$

$$C_2^{bb} = \frac{1}{\omega\Omega}\left[\frac{1}{\Omega}(Z_1 + Z_{-1} - Z_2 - Z_{-2}) + \frac{\partial}{\partial\Omega}(Z_1 + Z_{-1})\right], \tag{120}$$

$$\begin{aligned} C_3^{bb} = &\frac{1}{2\omega\Omega}\left\{\frac{1}{\Omega}(5Z_2 + 5Z_{-2} - 2Z_1 - 2Z_{-1} - 6Z_0) - \frac{\partial}{\partial\Omega}(Z_2 + Z_{-2} + 2Z_1 + 2Z_{-1})\right. \\ &-\frac{k_\varphi}{\Omega}\frac{\partial}{\partial k_\varphi}(Z_2 + Z_{-2} + 2Z_1 + 2Z_{-1} - 6Z_0) + \frac{1}{k_\varphi}\frac{\partial}{\partial k_\varphi}\left[(\omega+\Omega)\,\tilde{Z}_{-1} - (\omega-\Omega)\,\tilde{Z}_1\right] \\ &\left.+\frac{4}{\Omega^2}\left[(\omega - 2\Omega)Z_2 - (\omega + 2\Omega)Z_{-2} - 2(\omega-\Omega)Z_1 + 2(\omega+\Omega)Z_{-1}\right]\right\}, \end{aligned} \tag{121}$$

$$\begin{aligned} C_1^{nb} = &5A_{nn} + \frac{1}{2\omega\Omega}\left\{-\frac{\partial}{\partial\Omega}(Z_2 + Z_{-2} + 2Z_1 + 2Z_{-1}) + \frac{k_\varphi}{\Omega}\frac{\partial}{\partial k_\varphi}(4Z_0 - Z_1 - Z_{-1} - Z_2 - Z_{-2})\right. \\ &+\frac{4}{\Omega^2}\left[(\omega - 2\Omega)Z_2 - (\omega + 2\Omega)Z_{-2} - 2(\omega-\Omega)Z_1 + 2(\omega+\Omega)Z_{-1}\right] \\ &\left.+\frac{1}{k_\varphi}\frac{\partial}{\partial k_\varphi}\left[(\omega+\Omega)\,\tilde{Z}_{-1} - (\omega-\Omega)\,\tilde{Z}_1\right] + \frac{1}{k_\varphi^2}\left[(\omega-\Omega)\,\tilde{Z}_1 - (\omega+\Omega)\,\tilde{Z}_{-1}\right]\right\}, \end{aligned} \tag{122}$$

$$C_2^{nb} = \frac{1}{2}\left[-3B_{nb} + k_\varphi\frac{\partial}{\partial k_\varphi}B_{nb} + \frac{1}{\omega\Omega}\frac{\partial}{\partial\Omega}(Z_1 + Z_{-1})\right], \tag{123}$$

$$C_3^{nb} = \left\{\nabla'_n\left[-A_{nn} + \frac{1}{2\omega^2\Omega}(Z_1 - Z_{-1})\right]\right\} + \frac{b_p}{2\omega}\frac{\partial}{\partial k_\varphi}(\tilde{Z}_{-1} - \tilde{Z}_1), \tag{124}$$

$$C_{n\|} = -\frac{1}{b_p}\left\{\nabla'_n\left[\frac{B_{nb}}{2} - A_{\|\|} + \frac{1}{2\omega^2\Omega}(Z_{-1} - Z_1)\right]\right\}, \tag{125}$$

$$\begin{aligned} D_1^{nn} = &\frac{1}{2\omega\Omega}\left\{\frac{1}{k_\varphi}\frac{\partial}{\partial k_\varphi}\left[(\omega-\Omega)\,\tilde{Z}_1 + (\omega+\Omega)\,\tilde{Z}_{-1}\right] + \left(\frac{k_\varphi}{\Omega}\frac{\partial}{\partial k_\varphi} + \frac{\partial}{\partial\Omega}\right)(Z_2 - Z_{-2} + Z_1 - Z_{-1})\right. \\ &+\frac{4}{\Omega^2}\left[2\left\{(\omega-\Omega)Z_1 + (\omega+\Omega)Z_{-1} - \omega Z_0\right\} - (\omega-2\Omega)Z_2 - (\omega+2\Omega)Z_{-2}\right] \\ &\left.+\frac{5}{\Omega}(Z_{-2} - Z_2 + Z_1 - Z_{-1})\right\}, \end{aligned} \tag{126}$$

$$\begin{aligned} D_2^{nn} = &\frac{1}{2\omega\Omega}\left\{\frac{4}{\Omega^2}\left[(\omega-\Omega)Z_1 + (\omega+\Omega)Z_{-1} - 2\omega Z_0\right]\right. \\ &\left.+\left(2\frac{k_\varphi}{\Omega}\frac{\partial}{\partial k_\varphi} + \frac{\partial}{\partial\Omega} - \frac{1}{\Omega}\right)(Z_{-1} - Z_1)\right\}, \end{aligned} \tag{127}$$

$$\begin{aligned} D_3^{nn} = &\frac{1}{2\omega}\left\{\frac{1}{\Omega^2}\left[\nabla'_n(Z_2 - Z_{-2} - Z_1 + Z_{-1})\right] - \frac{1}{\omega\Omega}\left[\nabla'_n(Z_1 + Z_{-1})\right]\right. \\ &\left.+ b_p\frac{\partial}{\partial k_\varphi}(\tilde{Z}_1 + \tilde{Z}_{-1})\right\}, \end{aligned} \tag{128}$$

$$D_3^{bb} = D_3^{nn} + \frac{1}{\omega\Omega^2}\left[\nabla'_n(Z_1 - Z_{-1}))\right], \tag{129}$$

$$\begin{aligned} D_1^{\|\|} = &\frac{1}{\Omega k_\varphi}\left[\left(k_\varphi\frac{\partial^2}{\partial k_\varphi^2} + 2\frac{\partial}{\partial k_\varphi}\right)\tilde{Z}_0 + \omega\left(2 + k_\varphi\frac{\partial}{\partial k_\varphi}\right)B_{n\|}\right. \\ &\left.+ 2\Omega B_{b\|}\left(2\left(\frac{\omega}{\Omega}\right)^2 - 1\right)\right], \end{aligned} \tag{130}$$

$$D_2^{\|\|} = -\frac{1}{\omega\Omega}\left\{\nabla'_n\left[\frac{1}{\omega}\left(1+k_\varphi\frac{\partial}{\partial k_\varphi}\right)Z_0+\frac{1}{2\Omega}\left(1+k_\varphi\frac{\partial}{\partial k_\varphi}\right)(Z_{-1}-Z_1)\right]\right\} + \frac{b_p}{\omega}\frac{\partial}{\partial k_\varphi}\left(1+k_\varphi\frac{\partial}{\partial k_\varphi}\right)\tilde{Z}_0, \quad (131)$$

$$D_1^{nb} = \frac{1}{\omega\Omega}\left[\frac{1}{2\Omega}(Z_{-2}-Z_2)+\frac{\partial}{\partial\Omega}(Z_1-Z_{-1})\right], \quad (132)$$

$$D_2^{nb} = \frac{1}{\omega\Omega}\left[\frac{1}{\Omega}(Z_{-2}-Z_2+Z_1-Z_{-1})+\frac{\partial}{\partial\Omega}(Z_1-Z_{-1})\right], \quad (133)$$

$$D_{n\|} = \frac{1}{2\omega\Omega^2 b_p}\left\{\nabla'_n\left[k_\varphi\frac{\partial}{\partial k_\varphi}(Z_{-1}-Z_1)-\frac{\Omega}{\omega}(Z_1+Z_{-1})\right]\right\}, \quad (134)$$

$$G_{b\|} = \frac{1}{\Omega}\left\{\frac{1}{\omega k_\varphi}\left[\nabla_n(\omega_p^2-\omega\,\tilde{Z}_0)\right]-\beta_{\|b}\frac{\partial}{\partial k_\varphi}\tilde{Z}_0\right\}. \quad (135)$$

- Terms relative to the force-free current :

$$K = \frac{\lambda u}{\omega k_\varphi}\left(\frac{\omega_p}{v_{th}}\right)^2, \quad (136)$$

$$X_l = (2\lambda-1)(Y_l-Y_l^\lambda)-z_l^2\left(Y_l+\lambda\gamma^2Y_l^\lambda\right), \quad (137)$$

$$W_l = Y_l^\lambda\left[1+(\gamma z_l)^2\right]-Y_l, \quad (138)$$

$$L_1 = (\omega+\Omega)X_{-1}-(\omega-\Omega)X_1+4\Omega-\lambda\frac{\Omega}{\omega}\left[(\omega+\Omega)W_{-1}+(\omega-\Omega)W_1-\frac{2\omega}{\lambda}\right], \quad (139)$$

$$L_2 = 2\omega X_0-(\omega-\Omega)X_1-(\omega+\Omega)X_{-1}+\lambda\frac{\Omega}{\omega}\left[(\omega+\Omega)W_{-1}-(\omega-\Omega)W_1-\frac{2\omega}{\lambda}\right]. \quad (140)$$

As the tensor is established up to second order in the small parameters ϵ it is also of second order as a differential operator in the poloidal plane. If the imaginary parts of the dispersion functions are neglected, $\overleftrightarrow{\epsilon}$ is Hermitian with respect to the scalar product defined for vector fields

$$<\vec{F}|\vec{G}>=\int\vec{F}^\star\cdot\vec{G}\,d^3x. \quad (141)$$

This follows from the fact that the time-averaged power absorbed by the plasma is given by

$$P=\frac{1}{2}Re<\vec{j}|\vec{E}>=\frac{\epsilon_0\omega}{2}<\vec{E}|\overleftrightarrow{\epsilon}^a\vec{E}>, \quad (142)$$

where $\overleftrightarrow{\epsilon}^a = 1/2(\overleftrightarrow{\epsilon}-\overleftrightarrow{\epsilon}^\dagger)$ is the anti-Hermitian part of the tensor. In the absence of dissipation, represented by the imaginary parts of the dispersion functions, there is no power absorbed and the tensor must therefore be Hermitian. This property implies that the components of the tensor as operators on a scalar field must satisfy the relation

$$\epsilon_{\mu\nu} = \epsilon^{\dagger}_{\nu\mu} \qquad\qquad \mu,\nu \in \{n, b, \|\}, \tag{143}$$

with respect to the scalar product

$$< f \mid g >= \int f^{\star} g d^3x. \tag{144}$$

One can easily show that $\overleftrightarrow{\epsilon}$ is Hermitian to the order considered by checking condition (143) and using the adjoint forms of the operators in the poloidal plane

$$\begin{aligned} (\nabla_n)^{\dagger} &= -\nabla_n - (\nabla\cdot \vec{e}_n) = -\nabla_n + (\beta_{\|b} - \beta_{b\|}), \\ (\nabla_p)^{\dagger} &= -\nabla_p - (\nabla\cdot \vec{e}_p) = -\nabla_p + (\beta_{n\|} - \beta_{\|n}). \end{aligned} \tag{145}$$

V. Conclusions

The equations describing a stationary toroidal axisymmetric plasma have been derived from kinetic theory up to the first order in the small parameter ϵ_e. With the appropriate assumptions, we obtain the Grad-Shafranov equation of MHD. The dielectric tensor of such a plasma has then been derived up to the orders $\epsilon_e\epsilon_p$ and ϵ_p^2 in these small parameters. In the absence of dissipation the tensor is shown to be a Hermitian second order differential operator in the poloidal plane. It is therefore suitable for implementation in a numerical code based on a variational formulation.

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Appendix

We express the coefficients $\beta_{\mu\nu}, \mu, \nu \in \{n, b, \|\}$ defined by Eq.(9) in terms of the variables (ψ, χ, φ) :

$$\beta_{nn} = 0, \quad \beta_{nb} = -b_p \beta_{n\varphi}, \quad \beta_{n\|} = b_\varphi \beta_{n\varphi}, \tag{A.1}$$

$$\beta_{bn} = -b_p \beta_{\varphi n} - \nabla_p b_p, \quad \beta_{bb} = -b_p b_\varphi (\beta_{p\varphi} + \beta_{\varphi p}) + b_\varphi^2 \nabla_n \frac{b_p}{b_\varphi}, \quad \beta_{b\|} = b_\varphi^2 \beta_{p\varphi} - b_p^2 \beta_{\varphi p},$$

$$\beta_{\|n} = b_\varphi \beta_{\varphi n} + \nabla_p b_\varphi, \quad \beta_{\|b} = b_\varphi^2 \beta_{\varphi p} - b_p^2 \beta_{p\varphi}, \quad \beta_{\|\|} = b_p b_\varphi (\beta_{p\varphi} + \beta_{\varphi p}) + b_\varphi^2 \nabla_n \frac{b_p}{b_\varphi},$$

where

$$\begin{aligned} \beta_{n\varphi} &= \nabla_p \ln \mid \nabla\psi \mid, \qquad \beta_{p\varphi} = -\nabla_n \ln \mid \nabla\chi \mid, \\ \beta_{\varphi n} &= -\nabla_p \ln \mid \nabla\varphi \mid, \quad \beta_{\varphi p} = \nabla_n \ln \mid \nabla\varphi \mid . \end{aligned} \tag{A.2}$$

Using a Solovev[6] solution of Eq.(33), one can evaluate the leading orders of the different coefficients $\beta_{\mu\nu}$. To obtain this solution one sets the following profiles

$$\begin{aligned} I(\psi) &= const \\ p(\psi) &= a\psi \end{aligned} \tag{A.3}$$

In toroidal variables $(\varrho, \theta, \varphi)$, defined from the cylindrical variables by the following relations

$$\begin{aligned} r &= R + \varrho\cos\theta \\ z &= -\varrho\sin\theta, \end{aligned} \tag{A.4}$$

this solution, to first order in the small inverse aspect ratio ϱ/R and for a circular cross section of magnetic surfaces, is given by

$$\psi = -\frac{1}{4} a \mu_0 R^2 \varrho^2, \tag{A.5}$$

where R is the major radius of the magnetic axis.

The corresponding magnetic field can be written as

$$\vec{B} = \frac{R}{r} B_0 \left(\frac{\varrho}{R q_s} \vec{e}_p + \vec{e}_\varphi \right), \tag{A.6}$$

where q_s is the safety factor.

Evaluating relations (A.1) using Eqs.(A.5) and (A.6) yields the leading order of the coefficients :

$$\begin{aligned} \beta_{nn} = 0, \quad \beta_{n\|}, \beta_{\|n}, \beta_{\|b} = \mathcal{O}(\epsilon_e) \quad \beta_{b\|} = \mathcal{O}(\epsilon_p), \\ \beta_{nb}, \beta_{bn}, \beta_{bb} = \mathcal{O}(\epsilon_e^2), \quad \beta_{\|\|} = \mathcal{O}(\epsilon_e \epsilon_p). \end{aligned} \tag{A.7}$$

Note that the coefficients $\beta_{b||}$ and $\beta_{||\,||}$ have been weighted with the small parameter ϵ_p relative to the perturbation. This is due to the fact that the Solovev solution leads to the following results

$$\beta_{p\varphi} = \frac{1}{\varrho} = \mathcal{O}(\epsilon_p), \qquad \nabla_n b_p = \frac{b_p}{\varrho} = \mathcal{O}(\epsilon_e \epsilon_p), \tag{A.8}$$

as ϱ can be of the same order as the wavelength of the perturbation. Furthermore, $\beta_{b||}$ is the only non-vanishing coefficient in a straight cylindrical geometry, therefore the coefficients must be given the above weighting so as to recover the correct results in the corresponding limit. However we must realize that the stationary state solved to second order in ϵ_e would contain terms of the order of $\beta_{b||}\nabla_n F$, which consequently must be considered of order ϵ_e^2.

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