Semester project in mathematics, BA6

A primal-dual semidefinite programming approach to stochastic linear quadratic control problems

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Abstract

The purpose of this article, which is partly a closed review of the article *Stochastic linear-quadratic control via semidefinite programming* in SIAM Journal Control Optimization, Volume 40, No. 3, pages 801 to 823, written by Yao, Zang and Zhou, is to derive an algorithm for solving stochastic linear quadratic control problems over infinite time horizon using a primal-dual semidefinite programming approach.

We present briefly semidefinite programming problems formulation and the associated dual, we discuss when duality holds between both the primal and the dual and then give some methods to solve such problems and mention their complexity. In a second time, linear quadratic control problems will be discussed by first looking at the deterministic case and then the stochastic case.

We replace the traditional Riccati equation related to control problems by a semidefinte programming primal problem, its associated dual and a generalized version of the Riccati equation. Then, stability and optimality properties of the control found by the alternative methods are stated. Thereafter we can give an algorithm to solve stochastic linear quadratic control problems and show some application examples of the algorithm that has been developed.

This paper is ended by a short description of the S-procedure and an application example to control theory.

Contents

Introduction

This article is a closed review of (28) by Yao, Zang and Zhou and is meant to derive an algorithm for solving stochastic linear quadratic control problems over infinite time horizon using a primal-dual semidefinite programming approach.

In a first time semidefinite programming will be presented and in a second time, an application of semidefinite programming to the control theory is considered and finally a short description of the S-procedure is given.

In Chapter 2, we introduce semidefinte programming theory and the notations used in this paper. We refer for instance to Boyd and Vandenberghe (21) and Laurent and Rendl (10) for good overviews of semidefinite programming theory. Parallels between linear programming and semidefinte programming are drawn and a trial to understand the differences between both is done, for instance the differences in duality results. Some important results about duality in semidefinte programming are reviewed, see e.g. in the article by Monteiro (15) and two examples where the strong duality does not hold are given, see Todd (20) for more examples. The relation between strong duality and the Farkas lemma is given, lemma that is related to the S-procedure in Chapter 4. Then we close this chapter by giving some methods to solve semidefinite programming problems and briefly mention their complexity. We focus on three different methods, namely the ellipsoid method which has an historical importance (see e.g. Nemirovskiy and Yudin (16)), the Interior methods, which are the most used methods, and in particularly the primal-dual path following algorithms (see Monteiro (14), Helmberg, Rendl, Vanderbei and Wolkowicz (7) and Zhang (30)) and the subgradient methods and mention some closely related methods for large-scale semidefinite programming problems and eigenvalue problems (we refer to Helmberg and Rendl (6) for an overview).

In Chapter 3, we discuss linear quadratic control problems by first looking at the deterministic case and then the stochastic case. Some important definition such as stability and attainability of the control (problem) are given, see e.g YAO, ZANG and ZHOU (27; 28). The algorithm for continuous time linear quadratic control problems over infinite time horizon is recalled, we refer for instance to (9), which underlines the importance of the RICCATI equation in control theory.

The main part of our work is the stochastic case in Chapter 3. We give a formulation of the control problem and justify it by an example of problem statement drawn from the article by YAO, ZHANG and ZHOU (28). Then the RICCATI equation in the stochastic case is derived, knowing the nature of the control (open-loop or feedback) according to the method presented by LUO and FENG (11). After that, a brief discussion about when the RICCATI equation has certainly a solution or not depending on the cost matrices is hold, see for instance RAMI and ZHOU (18). Then, a semidefinte programming primal problem which corresponds to the RICCATI equation in a sense that its optimal values solve the RICCATI equation is given. We also give the corresponding dual and remark that both semidefinite programming problems avoid the difficulties that appear when solving the RICCATI equation. The notion of pseudo inverse of a matrix is then introduced and within a definition of the generalized RICCATI equation. Thereafter, we establish some stability results in relation to the solutions of the semidefinite programming problems, see e.g. RAMI and ZHOU (18). Then link between optimality property of the control and the results found by our alternative methods are established, we refer to YAO, ZANG and ZHOU (27; 28). According to the stability and optimality results, a systematic method to solve the control problem is given, namely we enunciate the algorithm in YAO, ZANG and ZHOU in (28). We continue with some simple examples of control problems, which we apply the algorithm to. Some of those examples can be found in (28) by YAO, ZANG and ZHOU. We mention briefly the ε-method in relation to a problem with no optimal attainable control, but still well-posed.

In Chapter 4, we give a small overview of the S-procedure and prove the S-procedure when it is lossless. We also establish the relation between FARKAS lemma and the S-procedure. Finally, an example of application of the S-procedure to control theory is given.

Semidefinite programming

2.1 Generalities

Depending on the authors, different notations to state a semidefinite programming problem can be found. We will use the following one for the primal problem

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(BX) \\ \text{subject to} & \mathbf{Tr}(A_i X) = c_i \quad i = 1, \ldots, m \\ & X = X^T \succeq 0. \end{array} \tag{PSDP}$$

Here the variable is the matrix $X = X^T \in S^n$.

The dual problem associated with the semidefinite programming problem (PSDP) is

$$\begin{array}{ll} \text{maximize} & c^T y \\ \text{subject to} & G(y) \preceq 0, \end{array} \tag{DSDP}$$

where

$$G(y) \triangleq \sum_{i=1}^{n} A_i y - B.$$

It is wished to minimize a linear function with respect to $y \in \mathbb{R}^m$ subject to matrix inequalities. From here on, we will consider that the problem data belong to the right spaces, namely c is a vector in $\mathbb{R}^m$ and $A_1, \ldots, A_m \in \mathbb{R}^{nxn}$ are $m + 1$ symmetric matrices with coefficients in $\mathbb{R}^{nxn}$. Further the set of symmetric matrices in $\mathbb{R}^{nxn}$ will denoted by S^n. The inequality sign in $G(y) \preceq 0$ means that $G(y)$ is negative semidefinite, i.e., $z^T G(y) z \leq 0$ for all $z \in \mathbb{R}^n$ and respectively $G(y) \prec 0$ means that $G(y)$ is negative definite. Semidefinite programming with symmetric matrices can be regarded as an extension of linear programming where the matrix inequalities replace the component-wise inequalities between vectors as follows.

Let us first write a linear programming dual problem (dLP) on standard form

$$\begin{array}{rl} \text{maximize} & c^T y \\ \text{subject to} & Ay \geq b. \end{array} \tag{dLP}$$

By setting $G(y) = \operatorname{diag}(Ay + b)$, i.e.,

$$B = \operatorname{diag}(b), \quad A_i = \operatorname{diag}(a_i), \quad i = 1, \ldots, m,$$

where $A = [a_1 ... a_m] \in \mathbb{R}^{nxm}$, we can express dLP as a semidefinite programming problem.

Therefore parallels between both the semidefinite programming and the linear programming theory can be established. Many algorithms for solving LP problems have generalizations for solving semidefinite programming problems. However, one big difference is that there is no simplex method for semidefinite programming (even if some cutting planes method can be compared to simplex method). There are some other important differences, for instance, duality results are weaker for semidefinite programming than for linear programming problems.

2.2 Duality results

Let us now look at the duality results for semidefinite programming.

Suppose that X is dual feasible and y is primal feasible. Then, using the fact that $\mathbf{Tr}(AB) \geq 0$ when $A = A^T \succeq 0$ and $B = B^T \succeq 0$, it follows that

$$\mathbf{Tr}(BX) - c^T y = \mathbf{Tr}(BX) - \sum_{i=1}^{m} \mathbf{Tr}(A_i X y_i) = -\mathbf{Tr}(XG(y)) \geq 0,$$

thus

$$c^T y \leq \mathbf{Tr}(BX).$$

This result is called weak duality, as it is in the linear programming theory.

But the strong duality result does not always hold for semidefinite programming, which is considerably different from linear programming.

Let us rewrite the primal semidefinite programming problem (PSDP)

$$\begin{array}{rl} \min & \mathbf{Tr}(CX) \\ \text{s.t.} & \mathbf{Tr}(A_i X) = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0. \end{array} \tag{2.2.1}$$

The only difference with an LP problem

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \tag{2.2.2}$$

is that $X \succeq 0$ replaces $x \geq 0$, which holds if and only if $u^T X u \geq 0$ for all u with $\|u\|_2 = 1$. So, to test the positive semidefinitness of a matrix, infinitely many cases have to be considered, but in LP problems there are only finitely many.

Let us look the following theorem to show more formally that strong duality of semidefinite problems does not always hold.

Theorem 2.1 (FARKAS' lemma)**.** *Let A be an $m \times n$ matrix and b an m–dimensional vector. Then exactly one of the following statements can hold at the time.*

1) *There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.*

2) *There exists $y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y < 0$.*

Proof. Let us take $x \geq 0$ such that $Ax = b$, then we have $y^T Ax = y^T b$, i.e. $(A^T y)^T x \geq 0$ and $(b^T y)^T \geq 0$ if $A^T y \geq 0$ and if $b^T y < 0$, $y^T Ax < 0$ and $A^T y < 0$ so if 1) holds 2) can not hold simultaneously.

Now consider the set

$$C = \{z \mid z = Ax, x \geq 0\},$$

which is closed and convex, where the closeness follows by considering basic feasible solutions. Then if

$\underline{b \in C;}$ Point 1) of Theorem 2.1 holds.

$\underline{b \notin C;}$ As C is a convex and closed set there exists $z^\star \in C$ such that $\|z^\star - b\|_2 \leq \|z - b\|_2 \forall z \in C$ and so it holds that

- $(z^\star - b)^T z \geq (z^\star - b)^T z^\star, \forall z \in C$ and
- $(z^\star - b)^T b < (z^\star - b)^T z^\star$.

Thus there exists y such that $y^T b < y^T z$ for all $z \in C$, so $y^T b < y^T z$, $x \geq 0$ and therefore $y^T A \geq 0$. Moreover, as $0 \in C$, $y^T b < 0$ and Theorem 2.1, Point 2) holds. □

We will see, in Chapter 4, that there exists a more general result called the S-procedure, see Corollary 4.2.

However, since

$$C = \{z \mid z_i = \mathbf{Tr}(A_i X), X \succeq 0\}$$

is not closed Theorem 2.1 does not always hold for semidefinite programming and therefore strong duality neither.

Strong duality holds if $\mathbf{Tr}(G(y)X) = 0$, since then

$$c^T y = \mathbf{Tr}(BX).$$

Using the fact that if A and B are symmetric positive semidefinite and $\mathbf{Tr}(AB) = 0$ then $AB = 0$, which can be shown by using the eigenvalue decomposition of A and considering separately the positive and zero eigenvalues, it turns out that if $\mathbf{Tr}(G(y)X) = \mathbf{Tr}(XG(y)) = 0$, since $G(y) \succeq 0$ and $X \succeq 0$, we have $XG(y) = 0$.

The condition $XG(y) = 0$ is referred to as the complementary slackness condition, in the sense that if the complementary slackness condition holds then the strong duality does as well.

Let us now look at two examples where strong duality does not hod.

Example 2.1. *(20, Page 17) In this example the duality gap is 0, but the primal problem optimal value is not attainable.*

The primal problem is stated as follows

$$\begin{aligned} \min \quad & y_1 \\ s.t \quad & \begin{pmatrix} y_1 & 1 \\ 1 & y_2 \end{pmatrix} \succeq 0. \end{aligned} \tag{2.2.3}$$

It has no optimal solution since $(\epsilon, \frac{1}{\epsilon})$ is feasible for all $\epsilon > 0$ but not for $\epsilon = 0$. The optimal value is therefore 0, but is never attained. The dual problem is

$$\begin{aligned} \min \quad & \boldsymbol{Tr}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X) \\ s.t \quad & \boldsymbol{Tr}(\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} X) = -1 \\ & \boldsymbol{Tr}(\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} X) = 0 \\ & X \succeq 0. \end{aligned} \tag{2.2.4}$$

It has one only feasible solution which is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ with an associated optimal value 0.

Example 2.2. *(22, Example 4.1.2.) In this example the duality gap is equal to a positive constant a that can be chosen arbitrarily big.*

The primal problem

$$\begin{aligned} \min \quad & C^T X \\ s.t. \quad & \boldsymbol{Tr}(A_1 X) = b_1 \\ & \boldsymbol{Tr}(A_2 X) = b_2 \\ & X \succeq 0, \end{aligned} \tag{2.2.5}$$

where $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $C = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *has* $X = \begin{pmatrix} 1 & 0 & x_{13} \\ 0 & 0 & 0 \\ x_{13} & 0 & x_{33} \end{pmatrix}$ *as solution with* x_{33} *positive. The resulting optimal value is therefore* a*, where* a *is any positive number.*

The associated dual problem is

$$\begin{array}{ll} \max & y_1 \\ s.t & \begin{pmatrix} a - y_1 & 0 & 0 \\ 0 & -y_2 & -y_1 \\ 0 & -y_1 & 0 \end{pmatrix} \succeq 0. \end{array} \tag{2.2.6}$$

The optimal solution is then $y = (0,0)^T$ *with optimal value* 0*, therefore the duality gap is* a*, which is positive.*

Recall the following theorem, which is a well known result about strong duality in semidefinite programming theory. We refer for example to Monteiro (15) for more details and proof.

Theorem 2.2. *Let us define the following sets which correspond respectively to the set of primal (PSDP) feasible (strictly feasible) and dual (DSDP) feasible (strictly feasible) solutions.*

- $F(P) = \{x \in \mathbb{R}^m \mid G(x) \succeq 0\}$,
- $F^0(P) = \{x \in F(P) \mid G(x) \succ 0\}$,
- $F(D) = \{Z \in S^n \mid X \succeq 0, \boldsymbol{Tr}(A_i Z) = c_i\}$,
- $F^0(D) = \{Z \in F(D) \mid Z \succ 0\}$.

Then the following hold where Val(P)*, respectively Val*(D)*, represents the optimal value of (PSDP), respectively (DSDP).*

1. *If Val*$(P) > -\infty$ *and* $F^0(P) \neq \emptyset$ *then the set of optimal solution for (DSDP) is non-empty, bounded and Val*$(P) =$ *Val*(D)*;*
2. *If Val*$(D) < \infty$ *and* $F^0(D) \neq \emptyset$ *then the set of optimal solution for (PSDP) is non-empty, bounded and Val*$(P) =$ *Val*(D)*;*
3. *If* $F^0(P) \neq \emptyset$ *and* $F^0(D) \neq \emptyset$ *then the set of optimal solution for (PSDP) and (DSDP) is non-empty, bounded and Val*$(P) =$ *Val*(D)*,i.e. strong duality holds.*

In linear programming the strong duality holds without restrictions.

2.3 Methods and problem complexity

There exist different methods to solve semidefinite programming problems, most of them being a generalization of the methods to solve linear programming problems. The principal class of methods to solve semidefinite programming are the interior methods. Another method we mention here is the ellipsoid method. Finally we will briefly talk about an alternative to interior methods, namely the subgradient methods.

It has been shown that semidefinite programming problem can be solved in polynomial time.

It is known that computing the eigenvalues of a matrix is an NP–hard problem, but a semidefinite programming problem can be solved in a polynomial time, which seems to be a contradiction. But, in reality, a semidefinite programming problem can be solved in polynomial time to a certain tolerance $\varepsilon > 0$ which is chosen on before hand. The complexity of interior methods to solve a semidefinite programming problem is of $O(\sqrt{n}\ln(1/\varepsilon))$ and therefore grows to infinity with ε getting smaller, what makes sense since the algorithms to get eigenvalues give them with a tolerance 0, see e.g. (22, chapter 10).

2.3.1 The ellipsoid method

We mention here the ellipsoid method for historical reason. It is the first method that could show that semidefinite programming problems can be solved in polynomial time. A cutting plane for the constraint set through any given infeasible point can be built, in polynomial time (see, e.g., (5)). One can therefore apply the ellipsoid method of Yudin and Nemirovsky (see (16)) to solve a semidefinite programming problem in polynomial time.

Even though semidefinite programming problems can be solved to any fixed prescribed precision in polynomial time, the running time for the ellipsoid method is really high in practice, which make us chose an other way of solving semidefinite programming problems. Actually the ellipsoid method has been replaced by the interior methods, which are faster.

2.3.2 Interior methods

Interior methods have some properties that make them particularly attractive. They have a practical efficiency in a way that for linear programming problems they are competitive with the simplex methods and they may be substantially faster when the number of variables and constraints is really large. As they have polynomial complexity, they have theoretical efficiency. They can also exploit the problem structure.

However the problem with interior methods is that with the number of operations in a problem, the size of the matrix grows a lot and then the efficiency of the methods is limited in the sense that the time to solve a problem is growing quite fast. For this reason people are still working on creating new methods which are basically interior ones, but that take into consideration the structure of the matrix to compute less operations. For some primal-dual path following algorithms, which are the class of interior methods considered here, we refer to MONTEIRO (14), HELMBERG, RENDL, VANDERBEI and WOLKOWICZ (7) and ZHANG (30) under others.

General algorithms

The difference in the interior methods for solving semidefinite programming compared to solving linear programming is that when a barrier transformation on

$$\begin{array}{ll} \min & \mathbf{Tr}(CX) \\ \text{s.t.} & \mathbf{Tr}(A_1 X) = b_i \quad i = 1, \dots, m \\ & X = X^T \succeq 0, \end{array}$$

is applied, it results that

$$\min\{\mathbf{Tr}(CX - \mu \ln(\det X))\}.$$

By the gradient of the barrier function $C - \mu X^{-1} - \sum_{i=1}^{m} A_i y_i = 0$. We define then $S = \mu X^{-1}$ which can be rewritten as $SX = \mu I$ or $XS = \mu I$. In linear programming case it makes no difference which of both equalities is chosen to continue, but as $XS \neq SX$ in general, it changes in semidefinite programming.

We will propose here the primal-dual path following algorithm exposed by MONTEIRO (14). First assume that $F^0(P) \times F^0(D) \neq \emptyset$, i.e the strong duality holds.

Algorithm

Step 0 Give $x^0 \in F^0(P)$ and $(S^0, y^0) \in F^0(D)$. Set $k = 0$.

Step 1 Set $x = x^k$, $(S, y) = (S^k, y^k)$ and choose $\mu = \frac{(\mathbf{Tr}(XS))}{n}$.

Step 2 Set $H = \sigma \mu I - X^{1/2} S X^{1/2}$, where $\sigma = \sigma_k \in [0, 1]$ is the centrality parameter and has to be chosen.

Step 3 Compute the search direction $(\Delta X, \Delta S, \Delta y)$ by solving the system

$$\begin{cases} X^{-1/2}(X\Delta S + \Delta X S)X^{1/2} + X^{1/2}(X\Delta S + \Delta X S)X^{-1/2} = 2H \\ \mathbf{Tr}(A_i \Delta X) = 0 \quad \forall i = 1, \dots, m \\ \displaystyle\sum_{i=1}^{m} \Delta y_i A_i + \Delta S = 0. \end{cases}$$

Step 4 Find the step length $\alpha = \alpha^k \geq 0$ such that $\widehat{X} = X + \alpha\Delta X$ is symmetric positive definite and $(\widehat{S}, \widehat{y}) = (S, y) + \alpha(\Delta S, \Delta y)$, with $\widehat{S}$ being symmetric positive definite.

Step 5 Let $X^{k+1} = \widehat{X}$, $(S^{k+1}, y^{k+1}) = (\widehat{S}, \widehat{y})$, replace k by $k+1$ and go to step 1.

In (14) they are using $H = \sigma\mu I - X^{1/2}SX^{1/2}$ for the condition $S = \mu X^{-1}$. Different authors present this algorithm slightly differently and the main difference is in the choice of the linearisation of $S = \mu X^{-1}$. For example, in (7), they linearise is as $XS = \mu I$.

2.3.3 Subgradient methods

The subgradient methods have the advantage over interior methods that they can solve problem of larger size. However they are less used since the algorithms are, in term of number of operations, relatively slow and moreover there exists no simple stopping criterion for subgradient methods that could guarantee a certain level of optimality.

General methods

In vector case, a subgradient to ϕ at u, where ϕ is a concave function on $\mathbb{R}^m$, is a vector $s \in \mathbb{R}^m$ such that

$$\phi(v) \leq \phi(u) + s^T(v - u) \quad \forall v \in \mathbb{R}^m.$$

Moreover if ϕ is differentiable at u, $\delta\phi(u)$ is the unique subgradient. It can be shown that if $\phi(u) = \min_{x \in X}\{f(x) - u^T g(x)\}$, then $-g(x(u))$ is a subgradient to ϕ at u, where $x(u)$ denotes an optimal solution to $\min_{x \in X}\{f(x) - u^T g(x)\}$. This result applied to (2.2.2) implies that

$$\phi(u) = \min_{x \in X}\{c^T x - u^T(Ax - b)\}$$

and $b - Ax(u)$ is a subgradient to ϕ at u. The general subgradient method follows

For a given $u^k \in \mathbb{R}^m$,

Step 1 Find x^k the optimal solution to $\min_{x \in X}\{c^T x - (u^k)^T(Ax - b)\}$

Step 2 Let the subgradient $s^k = b - Ax^k$

Step 3 Choose the step length θ^k, by default $\theta^k = 1/k$ is set.

Step 4 Update $u^{k+1} = u^k + \theta^k s^k$ and go to Step 1.

In matrix case, corresponding to semidefinite programming case, the methods follow the same steps, but the definition of subgradient is slightly different, namely a symmetric matrix G that satisfies

$$\mu(\tilde{P}) \geq \mu(P) + \mathbf{Tr}(G(\tilde{P} - P))$$

for any symmetric matrix $\tilde{P}$ is a subgradient of μ at P. For example the subgradient corresponding to $\mu(P) = \lambda_n(B - \sum_{i=1}^{n} A_i^T x)$ is $G = vv^t$, where v is the unitary eigenvector associated to the largest eigenvalue of $B - \sum_{i=1}^{n} A_i^T x$.

There exist some closely related methods for solving large-scale semidefinite programming problems and eigenvalue problems. In particular, we refer to (6) for an overview of spectral bundle method and prox-method.

Application of semidefinite programming to stochastic Linear-Quadratic control theory

3.1 Linear-Quadratic control problems

Let us first recall some definitions.

Definition (stabilizing)**.**

- An open-loop control $u(\cdot)$ is (mean-square) *stabilizing* at x_0 if $x(\cdot)$, the state corresponding to the system dynamics (3.1.1b) with initial state x_0, satisfies $\lim_{t\to\infty} \mathsf{E}\left[x(t)^T x(t)\right] = 0$.
- A feedback control $u(t) = Kx(t)$, where K is a constant matrix, is *stabilizing* if $\forall x_0$, it holds that $\lim_{t\to\infty} x(t) = 0$ where $x(\cdot)$ is the solution to the system dynamics (3.1.1b) with $u(t) = Kx(t)$.
- Accordingly, the system (3.1.1b) is (mean square) *stabilizable* if there exists a stabilizing feedback control on the form $u(t) = Kx(t)$.

Definition (attainable)**.**

- The control problem (3.1.1) is called *attainable* at x_0 if it is well-posed at x_0 and there exists an optimal admissible control, where
- (3.1.1) is called *well-posed* at x_0 if $-\infty < \inf_{u(\cdot)\text{ adm.}} J(x_0, u(\cdot)) < +\infty$.

We will consider the following linear-quadratic (LQ) control problem

$$J(x_0, u(\cdot)) = \min \int_0^\infty \left[x(t)^T Q x(t) + u(t)^T R u(t)\right] dt \tag{3.1.1a}$$

$$\text{s.t} \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{3.1.1b}$$

We suppose A, B, Q and R to be constant matrices through this paper and we require Q and R to be symmetric matrices. We also require that the control $u(\cdot)$ is square integrable on $\mathbb{R}^m$.

The following theorem is well known in optimal control theory.

Theorem 3.1. *Consider (3.1.1), where $Q \succeq 0$ and $R \succ 0$. Assume that (A, B) is controllable. Then the following holds*

i) $J^*(x_0) = x_0^T P x_0$, *where P is the unique symmetric positive definite solution to the* RICCATI *equation*

$$PA + A^T P + Q - PBR^{-1}B^T P = 0. \tag{3.1.2}$$

ii) $u^* = -R^{-1}B^T Px$ *is the optimal stabilizing feedback control.*

Proof. We apply the continuous time dynamic programming algorithm to (3.1.1). The ansatz is done that $V(x) = x^T Px$, which is a positive definite and radially unbounded function that solves the HAMILTON-JACOBI-BELLMAN equation, so the optimal control should be $u^* = -R^{-1}B^T Px$. Indeed

$$\begin{aligned}
&\min_u\{x^T Qx + u^T Ru + (x^T P + Px)(Ax + Bu)\}\\
&\quad= x^T Qx + (-R^{-1}B^T Px)^T R(-R^{-1}B^T Px)+\\
&\qquad (x^T P + Px)(Ax + B(-R^{-1}B^T Px))\\
&\quad= x^T Qx + x^T PBR^{-T}B^T Px+\\
&\qquad x^T(PA - PBR^{-1}B^T P)x + x^T(A^T P - PBR^{-1}B^T P)x\\
&\quad= x^T(Q + PA + A^T P - PBR^{-1}B^T P)x = 0,
\end{aligned}$$

with $u^* = -R^{-1}B^T Px$.

The assumption on controllability of (A, B) assures the existence of a unique positive definite solution to the RICCATI equation (3.1.2). □

We see in this theorem the importance of the RICCATI equation for LQ control theory. This theorem gives explicitly the form of the feedback control, but has an obvious drawback, namely the requirement that $R \succ 0$. Indeed, the cases when $R \succeq 0$ is on interest. YONG and ZHOU have proved in (29, Chapter 6, Proposition 2.4) that $R \succeq 0$ is a necessary condition for (3.1.1) to be well-posed.

3.2 Stochastic case

Analogously to the deterministic case, we will consider, in problem (3.2.1), A, B, C and D being constant matrices and Q, R being constant symmetric matrices. A one-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with $t \in [0, \infty)$ is introduced. Similarly the control will be denoted by $u(\cdot)$ and assumed to be square integrable on $\mathbb{R}^m$.

We will consider the following stochastic linear quadratic (SLQ) optimal control problem.

$$\min \quad \mathsf{E}\left[\int_0^\infty \left[x(t)^T Q x(t) + u(t)^T R u(t)\right] dt\right] \tag{3.2.1a}$$

$$\text{s.t} \quad \begin{cases} dx(t) = [Ax(t) + Bu(t)]\, dt + [Cx(t) + Du(t)]\, dW(t), \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{3.2.1b}$$

Let us first consider an example in order to motivate this problem formulation.

3.2.1 Example of a problem statement

The example we describe here can be found in (28). The problem statement we want to derive is

$$\begin{aligned} J(\begin{pmatrix} x(0) \\ y(0) \end{pmatrix}, u(\cdot)) = \\ \min \quad & \mathsf{E}\left[\int_0^\infty X(t)^T \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} X(t) dt\right] \\ \text{s.t} \quad dX(t) = & \left[\begin{pmatrix} r - 1/2\rho & 0 \\ 0 & b_I - 1/2\rho \end{pmatrix} X(t) + \begin{pmatrix} b - r \\ 0 \end{pmatrix} u(t)\right] dt \\ & + \left[\begin{pmatrix} 0 & 0 \\ 0 & \sigma_I \end{pmatrix} X(t) + \begin{pmatrix} \sigma \\ 0 \end{pmatrix} u(t)\right] dW(t) \\ X(0) = & \begin{pmatrix} z_0 \\ i_0 \end{pmatrix}. \end{aligned}$$

This problem corresponds to a one bound- and one stock- market.

The bound is governed by the price dynamics

$$dP_0(t) = rP_0(t)dt, \qquad P_0(0) = p_0;$$

similarly the stock is governed by

$$dP_1(t) = P_1(t)\left[bdt + \sigma dW(t)\right], \quad P_1(0) = p_1,$$

where $W(\cdot)$ is a one-dimensional standard Brownian motion. A wealth trajectory given by

$$dI(t) = I(t)\left[b_I dt + \sigma_I(t) dW(t)\right], \quad I(0) = i_0$$

is to be tracked by an agent with initial endowment z_0. The total wealth of the agent is denoted by $z(t)$ at any time $t \geq 0$. The market value of the stock is denoted by $\pi(t)$ and assumed to be traded continuously. Further it is assumed that there is no transaction cost, no withdraw for consumption and no payment of any dividend. Under those assumptions, we get the following condition on $z(t)$:

$$dz(t) = [rz(t) + (b-r)\pi(t)]\,dt + \sigma\pi(t)dW(t), \quad z(0) = z_0.$$

The agent task is to minimize

$$\mathsf{E}\left[\int_0^\infty e^{-\rho t} \mid z(t) - I(t) \mid^2 dt\right],$$

where ρ is the discount rate.

By choosing the state

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-1/2\rho t}\begin{pmatrix} z(t) \\ I(t) \end{pmatrix},$$

the objective function becomes

$$\min \mathsf{E}\left[\int_0^\infty \mid x(t) - y(t) \mid^2 dt\right] = \min \mathsf{E}\left[\int_0^\infty X(t)^T \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} X(t)dt\right],$$

which is what we wanted.

Let us define the control variable $u(t)$ as

$$u(t) = e^{-1/2\rho t}\pi(t),$$

then it holds that

$$\begin{aligned}
dX(t) &= \begin{pmatrix} dx(t) \\ dy(t) \end{pmatrix} = \begin{pmatrix} d\left(e^{-1/2\rho t}z(t)\right) \\ d\left(e^{-1/2\rho t}I(t)\right) \end{pmatrix} = \begin{pmatrix} -1/2\rho x(t)dt + e^{-1/2\rho t}dz(t) \\ -1/2\rho y(t)dt + e^{-1/2\rho t}dI(t) \end{pmatrix} \\
&= -1/2\rho \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} dt + \begin{pmatrix} re^{-1/2\rho t}z(t) + (b-r)e^{-1/2\rho t}\pi(t) \\ b_I e^{-1/2\rho t}I(t) \end{pmatrix} dt \\
&+ \begin{pmatrix} e^{-1/2\rho t}\sigma\pi(t) \\ e^{-1/2\rho t}\sigma_I(t)I(t) \end{pmatrix} dW(t) \\
&= \begin{pmatrix} [(r-1/2\rho)x(t) + (b-r)u(t)]\,dt \\ (b_I - 1/2\rho)y(t)dt \end{pmatrix} + \begin{pmatrix} \sigma u(t) \\ \sigma_I y(t) \end{pmatrix} dW(t) \\
&= \left[\begin{pmatrix} r-1/2\rho & 0 \\ 0 & b_I - 1/2\rho \end{pmatrix} X(t) + \begin{pmatrix} b-r \\ 0 \end{pmatrix} u(t)\right] dt \\
&+ \left[\begin{pmatrix} 0 & 0 \\ 0 & \sigma_I \end{pmatrix} X(t) + \begin{pmatrix} \sigma \\ 0 \end{pmatrix} u(t)\right] dW(t), \\
X(0) &= \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^0 \begin{pmatrix} z(0) \\ I(0) \end{pmatrix} = \begin{pmatrix} z_0 \\ i_0 \end{pmatrix}
\end{aligned}$$

So the problem is on the same form as (3.2.1).

Knowing if the optimal solution to (3.2.1) is an open-loop or a feedback control, we can determine the optimal control if there exists a feasible solution to the stochastic RICCATI equation.

From the control to the Riccati equation

We will now derive the Riccati equation from (3.2.1), the SLQ control problem, when supposing that there exists a feedback solution $u(t) = Kx(t)$. Let us first rewrite the problem with the assumption $u(t) = Kx(t)$.

The system dynamics changes as follow

$$\begin{cases} dx(t) = [A + BK]\, x(t)dt + [C + DK]\, x(t)dW(t) \\ x(0) = x_0, \end{cases}$$

where, introducing $X(t) = \mathsf{E}\left[x(t)x(t)^T\right]$, we get

$$\begin{cases} dX(t) = [A + BK]\, X(t) + X(t)\, [A + BK]^T \\ \qquad\qquad\qquad + [C + DK]\, X(t)\, [C + DK]^T \\ X(0) = X_0 = \begin{pmatrix} x_0 & x_0^T \end{pmatrix}. \end{cases}$$

The corresponding objective function (3.2.1a) changes to

$$J(K(\cdot)) = \int_0^\infty \mathbf{Tr}\ \left\{\left(Q + K^T RK\right) X(t)\right\} dt.$$

So the new problem becomes

$$\begin{aligned} &\min \int_0^\infty \mathbf{Tr}\ \left\{\left(Q + K^T RK\right) X(t)\right\} dt \\ &\text{s.t} \begin{cases} dX(t) = [A + BK]\, X(t) + X(t)\, [A + BK]^T \\ \qquad\qquad\qquad + [C + DK]\, X(t)\, [C + DK]^T \\ X(0) = X_0 = \begin{pmatrix} x_0 & x_0^T \end{pmatrix} \end{cases} \end{aligned} \tag{3.2.2}$$

Then, applying the dynamic programming algorithm for stochastic control problems over infinite time horizon, we get

1) Hamiltonian

$$\begin{aligned} H(X, u, P) &= f_0(X, u) + \mathrm{Tr}\ \left(f(X, u)P^T\right) \\ &= \mathrm{Tr}\ \left\{(Q + K^T RK)X + \left[(A + BK)X + X(A + BK)^T \right.\right. \\ &\qquad\qquad \left.\left. +\ (C + DK)X(C + DK)^T\right] P^T\right\} \end{aligned}$$

2) Pointwise minimization

$$\tilde{\mu} = \arg\min_K H(X, u, P)$$

3) Hamilton-Bellman-Jacobi equation

$$\begin{cases} 0 = -\frac{\partial}{\partial X^\star} H(X^*, u^*, P) \\ \frac{\partial H}{\partial u^\star}(X^*, u^*, P) = 0. \end{cases}$$

Since

- $\frac{\partial}{\partial X}\mathbf{Tr}(AXBX^T) = AXB + A^T XB^T$,
- $\frac{\partial}{\partial X}\mathbf{Tr}(AX) = A^T$ and
- $\frac{\partial}{\partial X}\mathbf{Tr}(AX^T) = A$,

we get

$$\begin{cases} 0 & = Q^T + K^T RK + P(A+BK) + (A+BK)^T P \\ & +(C+DK)P(C+DK)^T \\ \frac{\partial H}{\partial K} & = RKX^T + RKX + B^T PX^T + B^T P^T X + D^T PCX^T \\ & +D^T P^T CX + D^T PDKX^T + D^T PDKX = 0. \end{cases}$$

Using the fact that X and P are symmetric, it holds that

$$\begin{cases} 0 & = Q + K^T RK + P(A+BK) + (A+BK)^T P \\ & +(C+DK)P(C+DK)^T \\ \frac{\partial H}{\partial K} & = 2RKX + 2B^T PX + 2D^T PCX + 2D^T PDKX = 0 \end{cases}$$

and so

$$\begin{cases} 0 & = Q + K^T RK + P(A+BK) + (A+BK)^T P \\ & +(C+DK)P(C+DK)^T \\ \frac{\partial H}{\partial K} & = (R + D^T PD)K + B^T P + D^T PC = 0. \end{cases}$$

Thus we get

$$K = -(R + D^T PD)^{-1}(B^T P + D^T PC),$$

which, introduced in

$$Q + K^T RK + P(A+BK) + (A+BK)^T P + (C+DK)P(C+DK)^T = 0,$$

gives, after some simplifications,

$$\begin{aligned} &Q + PA + A^T P + C^T PC \\ &\qquad - (PB + C^T PD^T)(R + D^T PD)^{-1}(B^T P + D^T PC) = 0. \end{aligned} \tag{3.2.3}$$

This equation (3.2.3) is known as the stochastic RICCATI equation.

We refer to LUO and FENG in (11) for obtainning the RICCATI equation when starting with an open-loop control $u(t)$.

3.2.2 The Riccati equation

As it has been seen previously, the following equation is called the stochastic Riccati equation (or algebraic Riccati equation when C and D are 0)

$$\begin{aligned} &A^T P + PA + Q + C^T PC \\ &\qquad - (PB + C^T PD^T)(R + D^T PD)^{-1}(B^T P + D^T PC) = 0. \end{aligned} \tag{3.2.4}$$

If there exists a symmetric matrix P that solves (3.2.4) and such that $R + D^T PD \succ 0$, by analogy to the deterministic case it is known that the corresponding optimal feedback control is given by

$$u(t) = Kx(t), \text{ where } K = -(R + D^T PD)^{-1}(B^T P + D^T PC).$$

But it is almost never an easy task to solve the stochastic Riccati equation. In particular the inverse term $(R + D^T PD)$ is depending on the unknown, which does not help to solving the stochastic Riccati equation. Indeed there is no guarantee that $R + D^T PD \succ 0$, except, as in the deterministic case, when $R \succ 0$ and $Q \succeq 0$. It is well known that if $R \succ 0$ and $Q \succeq 0$ there exists a unique symmetric positive semidefinite solution P to the stochastic Riccati equation, see e.g. (18).

We already discussed the fact that in the deterministic case, the positive definiteness of R is a necessary condition for the linear quadratic problem (3.1.1) to have a solution.

In the case $D = 0$, the equation is slightly simplified and there exists a systematic way of solving it, namely by considering the associated Hamiltonian matrix. In the deterministic case ($C = D = 0$), the Hamiltonian matrix is given by $\begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix}$. We refer to (19, §2.7.2) for more details about the Hamiltonian matrix. We mention that this matrix is of big interest since none of its eigenvalues is pure imaginary if and only if there exists a solution to the Riccati equation. Applying a Routh-Hurwitz–type test is a way to verifying this condition.

We can replace the Riccati equation by a semidefinite programming problem.

From the Riccati equation to semidefinite programming

Indeed, in the deterministic case, the semidefinite programming problem

$$\begin{aligned} \max_P \quad & \mathbf{Tr}(P) \\ \text{s.t} \quad & \begin{pmatrix} R & B^T P \\ PB & Q + A^T P + PA \end{pmatrix} \succeq 0 \\ & P \in S^n \end{aligned} \tag{rSDP}$$

turns to have solutions that solves the RICCATI equation (3.1.2). The corresponding dual is given by

$$\begin{aligned} \min_{Z} \quad & \mathbf{Tr}(RZ_b + QZ_n) \\ \text{s.t} \quad & I + Z_u^T B^T + BZ_u + Z_n A^T + AZ_n = 0 \\ & Z := \begin{pmatrix} Z_b & Z_u \\ Z_u^T & Z_n \end{pmatrix} \succeq 0. \end{aligned} \tag{rDSDP}$$

It can be seen that the primal and the dual semidefinite programming problems are both well defined and the singularity of R and Q is no more a problem.

In a similar way, the semidefinite programming problem corresponding to the solution to the stochastic RICCATI equation (3.2.4) may be written as

$$\begin{aligned} \max_{P} \quad & \mathbf{Tr}(P) \\ \text{s.t.} \quad & \begin{pmatrix} R + D^T PD & B^T P + D^T PC \\ PB + C^T PD & Q + C^T PC + A^T P + AP \end{pmatrix} \succeq 0 \\ & P \in S^n. \end{aligned} \tag{srSDP}$$

with associated dual problem

$$\begin{aligned} \min_{Z} \quad & \mathbf{Tr}(RZ_b + QZ_n) \\ \text{s.t.} \quad & I + Z_u^T B^T + BZ_u + Z_n A^T + AZ_n \\ & \quad + CZ_n C^T + DZ_u C^T + CZ_u^T D^T + DZ_b D^T = 0 \\ & Z := \begin{pmatrix} Z_b & Z_u \\ Z_u^T & Z_n \end{pmatrix} \succeq 0. \end{aligned} \tag{srDSDP}$$

Until the end of this paper, we will assume that (srSDP) always has an attainable solution, since otherwise neither the semidefinite programming nor the RICCATI equation approach lead to a solution for the original stochastic linear quadratic problem (3.2.1). Those results are related to the S-procedure applied to control theory.

3.2.3 Stability

Here we will discuss the stability, which is a necessary requirement for the control to be admissible. The definitions of stabilizable controls given in the deterministic case can easily be extended to the stochastic case by changing (3.1.1b) to (3.2.1b). Moreover we will show that there is a close relation between stability results and the dual semidefinte programming problem. The aim is to prove the following theorem:

Theorem 3.2. *If a feasible solution,* $Z = \begin{pmatrix} Z_b & Z_u \\ Z_u^T & Z_n \end{pmatrix}$*, to the dual semidefinite programming problem* $(srDSDP)$ *is such that* $Z_n \succ 0$*, then the feedback control* $u(t) = Z_u Z_n^{-1} x(t)$ *is stabilizing.*

We will first recall the well known SCHUR's lemma as it is stated in (18).

Lemma 3.3 (SCHUR's lemma)**.** *Let the matrices* N*,* $M = M^T$ *and* $R = R^T \succ 0$ *be given with appropriate dimensions. Then the following conditions are equivalent:*

1. $M - NR^{-1}N^T \succeq 0$,
2. $\begin{pmatrix} M & N \\ N^T & R \end{pmatrix} \succeq 0$,
3. $\begin{pmatrix} R & N^T \\ N & M \end{pmatrix} \succeq 0$.

The non singularity condition on R is a strong condition and can be relaxed through the use of some generalized inverse matrices.

Definition (matrix pseudo-inverse)**.** We denote by M^+ the pseudo-inverse, referred as the MOORE-PENROSE inverse in some books, of the matrix M. The pseudo inverse has to fulfil the following conditions

$$MM^+M = M, M^+MM^+ = M^+, (MM^+)^T = MM^+, (M^+M)^T = M^+M.$$

Moreover if $M \succeq 0$, then $M^+ \succeq 0$, $(M^+)^T = M^+$ and $MM^+ = M^+M$. Hence the extended SCHUR's lemma is

Lemma 3.4 (extended SCHUR's lemma)**.** *Let the matrices* N*,* $M = M^T$ *and* $R = R^T$ *be given with appropriate dimensions. Then the following conditions are equivalent:*

1. $M - NR^+N^T \succeq 0$*,* $R \succeq 0$ *and* $N(I - RR^+) = 0$*,*
2. $\begin{pmatrix} M & N \\ N^T & R \end{pmatrix} \succeq 0$,
3. $\begin{pmatrix} R & N^T \\ N & M \end{pmatrix} \succeq 0$.

We can in a similar way define the stochastic generalized RICCATI equation as follows

$$\begin{aligned} A^TP + PA + Q + C^TPC \\ - (PB + C^TPD)(R + D^TPD)^+(B^TP + D^TPC) = 0. \end{aligned} \tag{3.2.5}$$

Two more results are needed in order to prove Theorem 3.2.

Theorem 3.5. *The following conditions are equivalent*

(i) System (3.2.1) *is mean-square stabilizable.*

(ii) Problem $(srDSDP)$ *is such that there exists a feasible* Z *with* $Z \succ 0$*, this condition is referred to as the* SLATER *condition in the literature.*

(iii) There exists a matrix K *and a symmetric matrix* Y *such that*

$$(A + BK)Y + Y(A + BK)^T + (C + DK)Y(C + DK)^T \prec 0, Y \succ 0.$$

In this case $u(t) = Kx(t)$ *is a stabilizing feedback control.*

(iv) There exists a matrix K *such that for any* X *there exists a unique solution* Y *to*

$$(A + BK)Y + Y(A + BK)^T + (C + DK)Y(C + DK)^T + X = 0,$$

Moreover if $X \succ 0$*, (respectively* $X \succeq 0$*) then* $Y \succ 0$*, (resp.* $Y \succeq 0$*) and the corresponding feedback control* $u(t) = Kx(t)$ *is stabilizing.*

(v) There exist a positive definite matrix Y *and a matrix* X *such that*

$$\begin{pmatrix} AY + YA^T + BX + X^TB^T & CY + DX \\ YC^T + X^TD^T & -Y \end{pmatrix} \prec 0.$$

The corresponding feedback control is $u(t) = XY^{-1}x(t)$ *and stabilizes.*

The proof of this result can be found, e.g, in (18, Theorem 1).

Theorem 3.6. *Let* P^* *be a feasible solution to* $(srSDP)$ *such that* P^* *is also a solution to the generalized* RICCATI *equation* (3.2.5)*. If the feedback control*

$$u(t) = -(R + D^TP^*D)^+(B^TP^* + D^TP^*C)x(t)$$

is stabilizing, then there exist complementary optimal solutions P^* *and* Z^* *to* $(srSDP)$ *and* $(srDSDP)$ *such that* $Z_n^* \succ 0$*, that is,* P^* *is optimal to* $(srSDP)$ *and there exists* Z^* *a dual complementary optimal solution with* $Z_n^* \succ 0$.

Proof. Since $u(t)$ is a stabilizing control Theorem 3.5 implies that

$$(A + BK)Y + Y(A + BK)^T + (C + DK)Y(C + DK)^T + I = 0 \quad (3.2.6)$$

has a symmetric positive definite solution $Y^* \succ 0$ (as $I \succ 0$), where

$$K = -(R + D^TP^*D)^+(B^TP^* + D^TP^*C).$$

Let us define

$$Z_n^* = Y^*, Z_u^* = KZ_n^* \text{ and } Z_b^* = K(Z_u^*)^T,$$

so that Z^* can be rewritten as

$$Z^* = \begin{pmatrix} Z_b^* & Z_u^* \\ (Z_u^*)^T & Z_n^* \end{pmatrix} = \begin{pmatrix} I & K \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & Z_n^* \end{pmatrix} \begin{pmatrix} I & 0 \\ K^T & I \end{pmatrix}.$$

Thus it can easily be seen that $Z \succeq 0$. Moreover since

$$\begin{aligned}
&I + (Z_u^*)^T B^T + B Z_u^* + Z_n^* A^T + A Z_n^* \\
&\quad + C Z_n^* C^T + D Z_u^* C^T + C (Z_u^*)^T D^T + D Z_b^* D^T = \\
&I + (Y^* K^T) B^T + B (K Y^*) + Y^* A^T + A Y^* \\
&\quad + C Y^* C^T + D (K Y^*) C^T + C (Y^* K^T) D^T + D (K Y^* K^T) D^T = \\
&I + (A + BK) Y^* + Y^* (A + BK)^T + (C + DK) Y^* (C + DK)^T = 0,
\end{aligned}$$

Z^* is a feasible solution to (srDSDP). It only remains to show that

$$\begin{pmatrix} R + D^T P^* D & B^T P^* + D^T P^* C \\ P^* B + C^T P^* D & Q + C^T P^* C + A^T P^* + A P^* \end{pmatrix} \begin{pmatrix} Z_b^* & Z_u^* \\ (Z_u^*)^T & Z_n^* \end{pmatrix} = 0$$

to establish that P^* and Z^* are complementary solutions since $Z_n^* = Y^* \succ 0$ by definition. Let us calculate

$$\begin{aligned}
&\begin{pmatrix} R + D^T P^* D & B^T P^* + D^T P^* C \\ P^* B + C^T P^* D & Q + C^T P^* C + A^T P^* + A P^* \end{pmatrix} \\
&= \begin{pmatrix} R + D^T P^* D & -(R + D^T P^* D) K \\ -K^T (R + D^T P^* D) & Q + C^T P^* C + A^T P^* + A P^* + S \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ -K^T & I \end{pmatrix} \begin{pmatrix} R + D^T P^* D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
S =& K^T (R + D^T P^* D) K \\
&- (P^* B + C^T P^* D)(R + D^T P^* D)^+ (B^T P^* + D^T P^* C) = 0.
\end{aligned}$$

Thus

$$\begin{aligned}
&\begin{pmatrix} R + D^T P^* D & B^T P^* + D^T P^* C \\ P^* B + C^T P^* D & Q + C^T P^* C + A^T P^* + A P^* \end{pmatrix} \begin{pmatrix} Z_b^* & Z_u^* \\ (Z_u^*)^T & Z_n^* \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ -K^T & I \end{pmatrix} \begin{pmatrix} R + D^T P^* D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix} \begin{pmatrix} Z_b^* & Z_u^* \\ (Z_u^*)^T & Z_n^* \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ -K^T & I \end{pmatrix} \begin{pmatrix} R + D^T P^* D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z_b^* - K (Z_u^*)^T & Z_u^* - K Z_n^* \\ (Z_u^*)^T & Z_n^* \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ -K^T & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.
\end{aligned}$$

$\square$

Let us now prove Theorem 3.2.

Proof. Since Z is feasible to (srDSDP) and $Z_n \succ 0$, by the SCHUR's lemma we get

$$Z_b \succeq Z_u Z_n^{-1} Z_u^T.$$

For the same reason,

$$I+(Z_u)^T B^T+BZ_u+Z_nA^T+AZ_n+CZ_nC^T+DZ_uC^T+CZ_u^TD^T+DZ_bD^T = 0,$$

so

$$\begin{aligned}
&(BZ_uZ_n^{-1}+A)Z_n \\
&\qquad + Z_n(A+BZ_uZ_n^{-1})^T + (C+DZ_uZ_n^{-1})Z_n(C+DZ_uZ_n^{-1})^T = \\
&AZ_n + BZ_u + Z_nA^T \\
&\qquad + Z_u^TB^T + CZ_nC^T + DZ_uC^T + CZ_u^TD^T + DZ_uZ_n^{-1}Z_u^TD^T \prec \\
&Z_u^TB^T + BZ_u + I + Z_nA^T \\
&\qquad + AZ_n + CZ_nC^T + DZ_uC^T + CZ_u^TD^T + DZ_uZ_n^{-1}Z_u^TD^T \preceq \\
&Z_u^TB^T + BZ_u + I + Z_nA^T \\
&\qquad + AZ_n + CZ_nC^T + DZ_uC^T + CZ_u^TD^T + DZ_bD^T = 0.
\end{aligned}$$

By Theorem 3.5(iii), with $Y = Z_n \succ 0$ and $K = Z_uZ_n^{-1}$, it holds that the control $u(t) = Kx(t) = Z_uZ_n^{-1}x(t)$ is stabilizing. □

3.2.4 Optimality

As was previously done for the stability of the controls, we want to establish the necessary and sufficient conditions on the semidefinite programming problem and the generalized RICCATI equation for the stochastic linear quadratic control problem (3.2.1) to be attainable. Recall that a control problem is attainable if there exists a corresponding admissible optimal control with finite corresponding optimal value.

Let us present some important results for optimality.

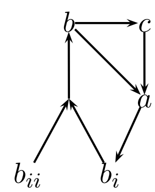
Theorem 3.7. *If the stochastic linear control problem (3.2.1) is attainable at any $x_0 \in \mathbb{R}^n$, then the corresponding semidefinite programming problem (srSDP) has an optimal solution that satisfies the generalized* RICCATI *equation (3.2.5).*

Theorem 3.8. *Let us denote by P^* any feasible solution to (srSDP). Then, if P^* satisfies the generalized* RICCATI *equation (3.2.5) and if the associated feedback control $u^*(t) = -(R + D^TP^*D)^+(B^TP^* + D^TP^*C)x^*(t)$ is stabilizing, it follows that $u^*(t)$ must be optimal to (3.2.1).*

Theorem 3.9. *Suppose that there exists complementary optimal solutions P^* and Z^* to (srSDP) and (srDSDP), then P^* satisfies the generalized* RICCATI *equation. Moreover, if $Z_n^* \succ 0$, then the stochastic linear quadratic control problem (3.2.1) has an attainable optimal feedback solution given by $u^*(t) = Z_u^*(Z_n^*)^{-1}x^*(t)$.*

The proofs of the three previous theorems given by YAO, ZHANG and ZHOU in (28) are complete and we estimate that it is not necessary for this article to add some more comments on those.

Theorem 3.10. *The following schema represents the existing links, where*



- (a) *corresponds to (3.2.1) is attainable at any $x_0 \in \mathbb{R}^n$.*
- (b) *corresponds to (srSDP) has an optimal solution P^* satisfying*

  (b_i) *P^* is a solution to the generalized* RICCATI *equation (3.2.5)*

  (b_{ii}) *$u(t) = -(R + D^T P^* D)^+(B^T P^* + D^T P^* C)x^*(t)$, the corresponding control is stabilizing.*
- (c) *corresponds to there exists P^* and Z^* with $Z_n^* \succ 0$ complementary optimal solutions to (srSDP) and (srDSDP).*

Remark that $(b) \implies (c)$ is given by Theorem 3.2. The other implications follow from Theorem 3.7 for $(a) \implies (b_i)$, Theorem 3.8 for $(b) \implies (a)$ and Theorem 3.9 for $(c) \implies (a)$.

Theorem 3.10, which establishes the links between the optimality results of the linear quadratic control, semidefinite programming problems and the generalized RICCATI equation underlines once more the importance of the stability results on the controls. Indeed if (b_{ii}) holds, i.e. the control is stabilizing, then all the conditions are equivalent.

We also see that both (b) and (c) imply that there exists an optimal admissible solution to (3.2.1) with finite corresponding objective value, so we have two ways to solve (3.2.1). However it is preferable to solve the control problem by semidefinte programming corresponding to (c) than to check the stability of the control in (b_{ii}), since there exist really efficient semidefinite programming solvers, most of them based on primal-dual interior point methods. Indeed if there is any dual optimal solution to the semidefinite programming problem with $Z_n^* \succ 0$, then the solver will return this solution. Hence verifying (c) corresponds to checking if $Z_n^* \succ 0$, which is easy compared to the (b_{ii}) condition. One more relevant observation is that even if (b) and (c) both give explicitly an optimal feedback control, they are not always the same since (b) implies (c), but the converse does not hold and therefore finding an optimal control in (c) does not imply the stability in (b).

3.2.5 General algorithm

Now that we have discussed the links between the optimality of the stochastic linear quadratic control problem and the semidefinite programming or generalized RICCATI equation approach, the following algorithm to solving the SLQ control problem (3.2.1) is obtainned. The following algorithm can be found in (28).

Step 1 Check if the feasible set of (srSDP) is non-empty.

If not **STOP**: the LQ problem cannot be solved by either the SDP approach or by the RICCATI equation.

Else continue.

Step 2 Check if there exists Z a feasible solution to (srDSDP) such that $Z \succ 0$.

If not **STOP**: the LQ problem is not mean-square stabilizable according to Theorem 3.5 and hence ill posed.

Else continue.

Step 3 We known that (srSDP) has an optimal solution, as (srSDP) is feasible and there exist a strictly feasible solution to (srDSDP). Check if any of the optimal solutions of (srSDP) satisfies the generalized RICCATI equation (3.2.5).

If not **STOP**: the LQ problem has no attainable optimal feedback control, see Theorem 3.7.

Else continue.

Step 4 Check if the control $u^*(t) = -(R + D^T P^* D)^+(B^T P^* + D^T P^* C)x^*(t)$ is stabilizing by using one the conditions in Theorem 3.5.

If yes **STOP**: $u^*(t)$ is the optimal control.

Else continue.

Step 5 Check if P^* and Z^* the optimal solutions to (srSDP) and (srDSDP) are complementary with $Z_n^* \succ 0$.

If yes **STOP**: $u^*(t) = Z_u^*(Z_n^*)^{-1}x^*(t)$ is the optimal control.

Else the (3.2.1) cannot be solved by any existing method.

We discussed previously that checking the condition in Step 5 was easier than checking the one in Step 4 for example, so in practical it can be useful to verify first condition in Step 5. If the result is positive, then the other steps are useless. If the test fails, then we should still start from Step 1 to know what is wrong with the problem, at which Step it first fails.

3.2.6 Examples

Let us first consider a simple deterministic example which is known to have no attainable control.

Example 3.1. *Consider the following LQ problem:*

$$\begin{aligned} \min \quad & \int_0^\infty x^2(t)dt \\ s.t. \quad & dx(t) = (-x(t) + u(t))dt \\ & x(0) = 1. \end{aligned}$$

We can identify $m = n = 1$, $A = -1$, $B = 1$, $C = D = 0$, $Q = 1$ *and* $R = 0$. *Let us first check the condition on the feedback control* $u(t) = kx(t)$ *to be stabilizing.*

$$dx(t) = (k-1)xdt, \text{ so } x(t) = x(0)e^{(k-1)t} = e^{(k-1)t}.$$

Thus

$$\lim_{t\to\infty} x(t) = 0 \text{ if and only if } k < 1.$$

Then the corresponding cost will be

$$\int_0^\infty x^2(t)dt = \int_0^\infty e^{2(k-1)t}dt = \frac{-1}{2(k-1)}.$$

Moreover we see that $\lim_{k\to-\infty} \frac{-1}{2(k-1)} = 0$, *so the optimal cost is zero, but is never attained since the corresponding control would be* $-\infty$.

Now apply the algorithm in order to verify that we get the same result. The primal SDP problem is

$$\begin{aligned} \max \quad & p \\ s.t. \quad & \begin{pmatrix} 0 & p \\ p & 1-2p \end{pmatrix} \succeq 0. \end{aligned}$$

There exists a unique feasible solution to the problem, namely $p^* = 0$. *The corresponding dual is*

$$\begin{aligned} \min \quad & z_n \\ s.t. \quad & 1 + 2z_u - 2z_n = 0 \\ & \begin{pmatrix} z_b & z_u \\ z_u & z_n \end{pmatrix} \succeq 0. \end{aligned}$$

For example the solution $z_b = 1$, $z_u = 0$ and $z_n = 1/2$ verifies the condition in Step 2. Let us now check the generalized Riccati *equation $1-2p-p^2 0^+ = 0$; it has no solution. Hence the algorithm stops and we know that there is no attainable feedback control.*

Let us now extend the previous example to a stochastic LQ control problem.

Example 3.2. *Consider the following LQ problem:*

$$\begin{aligned}
\min \quad & \int_0^\infty x^2(t)dt \\
s.t. \quad & dx(t) = (-x(t) + u(t))dt + (-x(t) + u(t))dW(t) \\
& x(0) = 1.
\end{aligned}$$

We can identify $m = n = 1$, $A = C = -1$, $B = D = 1$, $Q = 1$ and $R = 0$. Let us first check the condition on the feedback control $u(t) = kx(t)$ to be stabilizing.

$$dx(t) = (k-1)x(dt + dW(t)),$$

applying the Itô *formula, we get*

$$\begin{aligned}
dx^2(t) &= (k^2-1)x^2(t)dt + 2(k-1)x^2 dW(t), \\
d\mathsf{E}\left[x^2(t)\right] &= (k^2-1)\mathsf{E}\left[x^2(t)\right]dt + 0.
\end{aligned}$$

So

$$\mathsf{E}\left[x^2(t)\right] = x(0)e^{(k^2-1)t} = e^{(k^2-1)t}.$$

Thus

$$\lim_{t\to\infty} \mathsf{E}\left[x^2(t)\right] = 0 \textit{ if and only if } \mid k \mid < 1.$$

Then the corresponding cost will be

$$\mathsf{E}\left[\int_0^\infty x^2(t)dt\right] = \mathsf{E}\left[\int_0^\infty e^{(k^2-1)t}dt\right] = \frac{-1}{k^2-1}.$$

Moreover, we see that the minimum is attained when $k = 0$, so the optimal control is expected to be $u(t) = 0$.

Let us now apply the algorithm in order to verify our assumption. The primal SDP problem is

$$\begin{aligned}
\max \quad & p \\
s.t. \quad & \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix} \succeq 0.
\end{aligned}$$

The feasible solution to the problem are p's such that $0 \leq p \leq 1$. *We remark that the optimal value is given by* $p^* = 1$. *The corresponding dual is*

$$\begin{array}{ll} \min & z_n \\ s.t. & 1 + z_b - z_n = 0 \\ & \begin{pmatrix} z_b & z_u \\ z_u & z_n \end{pmatrix} \succeq 0. \end{array}$$

All solutions $z_b > 0$, z_u *such that* $z_u^2 \leq z_b(1+z_b)$ *and* $z_n = 1 + z_b$ *verify the condition in Step 2. Let us now check the generalized* Riccati *equation* $1 - p = 0$*; it has a solution at* $p^* = 1$, *which is also optimal to the primal SDP. Let us now verify if the control given by* $u(t) = -(R + Dp^*D)^+(Bp^* + Dp^*C)x(t)$ *is stabilizable. We get* $u(t) = 0x(t) = 0$, *which is stabilizable as it has been seen previously. Even if the algorithm is finished, let us look at the control given in Step 5. We know that* $p^* = 1$ *is optimal to the dual, we search for the complementary solution* z^* *on the form* $(z_b, z_u, z_n = 1 + z_b)$. *The condition*

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_b & z_u \\ z_u & z_b + 1 \end{pmatrix} = 0$$

implies that the optimal z^* *is given by* $(z_b^* = 0, z_u^* = 0, z_n^* = 1)$. *The control is then given by* $u(t) = z_u^*(z_n^*)^{-1}x(t)$, *so* $u(t) = 0$.

The same example as Example 3.2 with $R = -1$ instead of $R = 0$ is treated in (28). It interesting to remark that when $R = -1$, then any z^* on the form $(z_b, z_u, z_n = 1 + z_b)$ with $z_b > 0$ and $z_u^2 \leq z_b(1 + z_b)$ is optimal and complementary to the primal optimal solution $p^* = 1$. So the corresponding control is $u(t) = z_u^*(z_n^*)^{-1}x(t) = kx(t)$, and k satisfies $\mid k \mid < 1$. So in this case there exist infinitely many different optimal stabilizing controls, all having the same associate cost x_0^2.

We will now consider the one of the examples in (28), which is an example where the optimal solution to the primal SDP problem satisfies the generalized Riccati equation, but the corresponding control is not stabilizing.

Example 3.3. *Let us consider the problem*

$$\begin{array}{ll} \min & \mathsf{E}\left[\int_0^\infty (4x^2(t) - u^2(t))dt\right] \\ s.t. & dx(t) = u(t)dt + u(t)dW(t) \\ & x(0) = x_0. \end{array}$$

We can easily identify $m = n = 1$, $A = C = 0$, $B = D = 1$, $Q = 4$ *and* $R = -1$. *Suppose that there exists a feedback control* $u(t) = kx(t)$. *Let us* *derive the condition on k for this control to be stabilizing. Applying* Itô*'s formula, we get*

$$dx^2(t) = (k^2+2)x^2(t)dt + 2kx^2(t)dW(t),$$

so

$$d\mathsf{E}\left[x^2(t)\right] = (k^2+2)\mathsf{E}\left[x^2(t)\right]dt + 0$$

and therefore

$$\mathsf{E}\left[x^2(t)\right] = x_0 e^{(k^2+2)t}.$$

Hence we have stability when $k^2+2<0$, that is, when $-2<k<0$. Since

$$\mathsf{E}\left[\int_0^\infty x^2(t)dt\right] = \mathsf{E}\left[\int_0^\infty e^{(k^2+2k)t}dt\right] = \frac{-x_0^2}{k^2+2k},$$

the optimal cost will be

$$\mathsf{E}\left[\int_0^\infty (4x^2(t)-u^2(t))dt\right] = (4-k^2)\frac{-x_0^2}{k^2+2k} = (1-\frac{2}{k})x_0^2.$$

We see that the minimal cost is $2x_0$, but can not be attained when $x_0 \neq 0$ since the corresponding control $u(t)=2x(t)$ is not stabilizing. Let us now apply the previous algorithm. First the primal SDP problem is

$$\begin{array}{ll} \max & p \\ s.t. & \begin{pmatrix} -1+p & p \\ p & 4 \end{pmatrix} \succeq 0. \end{array}$$

It has only one feasible, thus optimal, solution, namely $p^* = 2$. Let us now look at the SDP dual

$$\begin{array}{ll} \min & 4z_n - z_b \\ s.t. & 1 + 2z_u + z_b = 0 \\ & \begin{pmatrix} z_b & z_u \\ z_u & z_n \end{pmatrix} \succeq 0. \end{array}$$

It has for instance the solution $(z_b = 3, z_u = -2, z_n = 2)$ that satisfies the condition in Step 2. Let us now verify if the optimal solution of the primal SDP problem is also a solution to the generalized Riccati *equation. Indeed*

$$4 - \frac{(2)^2}{-1+(2)} = 0.$$

Hence the corresponding control is given by $u(t) = \frac{-p^\star}{p^\star-1}x(t) = -2x(t)$, but we have already seen that this control is not stabilizing, so we should check the fifth Step. We look for an optimal solution to the dual z which is complementary to the primal optimal solution $p^* = 2$, so such that $4z_n - z_b = 2$ in *addition to the dual strict feasibility. This requirement is not possible since a solution* $(z_b = -1 - 2z_u, z_u, z_n = \frac{1-2z_u}{4})$ *is such that* $z_b z_n - z_u^2 = -1/4 < 0$*, so* $z \not\succeq 0$*. So the algorithm gives us no optimal control and the LQ control problem can not be solved by any method.*

We notice that in this example the stochastic LQ problem is well-posed, but our algorithm fails since we can't find any attainable optimal control to the problem. In (28), they reference one more example where the opposite happens, namely that the corresponding control to the optimal solution p^* of the SDP problem is mean-square stabilizable, but p^* is not a solution to the generalized RICCATI equation.

Facing a problem such as Example 3.3, we would like to find an optimal control as the problem is known t be well posed. In this case, YAO, ZHANG and ZHOU in (28) propose to make a so called "ε-approximation.

ε-approximation

The method consists of looking at the problem (LQ_ε) with $\varepsilon > 0$ instead of (3.2.1), where (LQ_ε) is obtainned by setting $R_\varepsilon = R + \varepsilon I$, $Q_\varepsilon = Q + \varepsilon I$ and keeping the other data unchanged. Then we compute the optimal P^* corresponding to the new primal SDP. With ε tending to 0, we get the same result, but the ε–depending corresponding control is optimal.

Let us now look at Example 3.3 again and apply the "ε–approximation" method.

Example 3.4. *The new primal SDP becomes*

$$\begin{aligned} &\max \quad p \\ &s.t. \quad \begin{pmatrix} -1+p+\varepsilon & p \\ p & 4+\varepsilon \end{pmatrix} \succeq 0. \end{aligned}$$

It has only one feasible, thus optimal, solution, namely

$$p_\varepsilon = \frac{4+\varepsilon+\sqrt{(4+\varepsilon)^2 - 4(4+\varepsilon)(1-\varepsilon)}}{2} = 2 + \frac{\varepsilon + \sqrt{5\varepsilon^2 + 20\varepsilon}}{2}.$$

So $p_\varepsilon = 2 + O(\sqrt{\varepsilon})$ *is such that the optimal cost converges to* $2x_0$ *when* ε *goes to* 0*.*

CHAPTER 4

The S-procedure

We will now talk about the S-procedure. Historically, the S-procedure has first been given by LURE and POSTNIKOV (12) in 1944, but they did not give any well-founded theory behind the method. The first who developed the theoretical background, the justification, of the S-procedure was YAKUBOVICH (23) and his students in 1971.

The next notable step in the theory of the S-procedure is due to TREIL and MEGRETSKY (13), who extended the results to infinite-dimensional spaces. Since then people mostly discuss new application of the S-procedure than give new theoretical extensions in their articles.

We can note that historically the term S-procedure comes from the appellation S-method that has been used by AIZERMAN and GANTMACHER (2, pp. 20-34) in their book. Indeed, in their description of the method, an auxiliary matrix S was introduced, where S stands for stability. Later the name has been changed to S-procedure or S-lemma.

According to the article by DERINKUYU and PINAR (4), the S-lemma is a special case, it is the name given to the S-procedure when the constraints consist of a single quadratic function, but in the paper written by PÓLICK and TERLAKY (17), the term S-procedure represents the method and the one S-lemma refers to the equivalence result of both conditions. In this paper, we will only use S-procedure without considering those distinctions.

Let $\sigma_k \colon V \to \mathbb{R}$, $k = 0, 1, \ldots, N$ be real real valued functionals defined on linear vector space V and consider the two conditions S_1 and S_2 as follows.

S_1: $\sigma_0(y) \geq 0$ for all $y \in V$ such that $\sigma_k(y) \geq 0$ for all k.

S_2: There exist $\tau_k \geq 0$ for each k such that

$$\sigma_0(y) - \sum_{k=1}^{N} \tau_k \sigma_k(y) \geq 0, \quad \forall y \in V.$$

The method consisting of verifying S_1 by use of S_2 is called the S-procedure. It can easily be seen that S_2 implies S_1. Indeed,

$$\sigma_0(y) \geq \sum_{k=1}^{N} \tau_k \sigma_k(y) \geq 0, \text{ when } \sigma_k(y) \geq 0.$$

Moreover, S_2 is generally a lot much easier to verify than S_1. There exist some special cases where S_1 and S_2 are equivalent. According to the denomination used by JÖNSSON (8) and PINAR and DERINKUYU (4) those cases are referred to as lossless S-procedure.

Let us now recall the definition of a regular function.

Definition (regular)**.** Let $\sigma_k \colon V \to \mathbb{R}$ be a function. The constraint $\sigma_k(y)$, verifying $\sigma_k(y) \geq 0$, for $k = 1, \ldots, N$ is said to be *regular* if there exists $y^* \in V$ such that $\sigma_k(y^*) > 0$.

We now introduce a notation we will need later.

Definition (convex polytope)**.** The *convex polytope*, C, with vertices at $x_1, \ldots, x_n \in \mathbb{R}^m$ is defined as the convex hull of these points, i.e.,

$$C = \mathrm{co}\{x_1, \ldots, x_n\} = \{\sum_{i=1}^{n} \alpha_i x_i \mid \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\}.$$

With these definitions, we can now give an example, due to YAKUBOVICH, of a case where the S-procedure is lossless.

Theorem 4.1 (YAKUBOVICH)**.** *Let $\sigma_k \colon V \to \mathbb{R}$, $k = 0, 1, \ldots, N$ and assume that the constraint $\sigma_k(y) \geq 0$ for all k is regular. Finally define the sets*

$$\mathfrak{K} = \{(\sigma_0(y), \sigma_1(y), \ldots, \sigma_n(y)) \mid y \in V\}$$

and

$$\mathfrak{N} = \{(n_0, n_1, \ldots, n_N) \mid n_0 < 0, n_k > 0, k = 1, \ldots, N\}.$$

If $(\mathfrak{K} \cap \mathfrak{N} = \emptyset)$ implies $(co(\mathfrak{K}) \cap \mathfrak{N} = \emptyset)$, then the S-procedure is lossless.

Proof. We refer to the article written by JÖNSSON (8) for the proof of this result. □

Corollary 4.2. *It follows immediately from this that if σ_k is a linear function on $\mathbb{R}^m$ and is regular, then the S-procedure is lossless for any $N < \infty$.*

Proof. In the linear case $\mathfrak{K}$ is convex, so $\mathfrak{K} = \mathrm{co}(\mathfrak{K})$. □

This linear case where $\sigma_k(y) = s_k^T y + r_k$, $s_k \in \mathbb{R}^m$ and $r_k \in \mathbb{R}$ is just a version of FARKAS Lemma, see 2.1. Indeed, let us set

$$A = \begin{pmatrix} s_1^T \\ \vdots \\ s_N^T \end{pmatrix}, \; B = \begin{pmatrix} -r_1 \\ \vdots \\ -r_N \end{pmatrix}, \; c = s_0^T \text{ and } d = -r_0,$$

then the losslessness of the S-procedure implies that F_1 and F_2 are equivalent, where

F_1: $cy \geq d$ for all y such that $Ay \geq B$,

F_2: $\exists \tau = (\tau_1, \ldots, \tau_N), \tau_k \geq 0$ such that $\tau A = c$ and $\tau B + d \leq 0$,

which is the FARKAS Lemma.

In this paper, we will mostly consider the special case with $V = \mathbb{R}^m$ and $\sigma_k(y) = y^T Q_k y + 2s_k^T y + r_k$, $k = 0, 1, \ldots, N$ where $Q_k \in S^m$, $s_k \in \mathbb{R}^m$ and $r_k \in \mathbb{R}$, that is the functionals σ_k are quadratic over $\mathbb{R}^m$. In this case, the condition S_2 corresponds to a linear matrix inequality (semidefinite programming) condition, namely S_2 is equivalent to

$$\exists \tau_k \geq 0 \quad \text{such that} \quad \begin{pmatrix} Q_0 & s_0 \\ s_0^T & r_0 \end{pmatrix} + \sum_{k=1}^{N} \tau_k \begin{pmatrix} Q_k & s_k \\ s_k^T & r_k \end{pmatrix} \succeq 0.$$

Now we will focus on a case where the S-procedure is lossless.

Theorem 4.3. *The S-procedure is lossless in the case of one quadratic constraint, in more details.*

If

$$\sigma_1(y) = y^T Q_1 y + 2s_1^T y + r_1 \geq 0$$

is regular, then S_1' and S_2' are equivalent, where

S_1': $y^T Q_0 y + 2s_0^T y + r_0 \geq 0$ *for all* $y \in \mathbb{R}^m$ *such that* $y^T Q_1 y + 2s_1^T y + r_1 \geq 0$.

S_2': $\exists \tau \geq 0$ *such that* $\begin{pmatrix} Q_0 & s_0 \\ s_0^T & r_0 \end{pmatrix} - \tau \begin{pmatrix} Q_1 & s_1 \\ s_1^T & r_1 \end{pmatrix} \succeq 0.$

We first want to rewrite this result in an equivalent form to make it easier to show.

Theorem. *Let A_1, $A_2 \in S^m$, b_1, $b_2 \in \mathbb{R}^m$ and c_1, $c_2 \in \mathbb{R}$. If there exists $\widehat{x} \in \mathbb{R}^m$ such that*

$$\widehat{x}^T A_2 \widehat{x} + 2b_2^T \widehat{x} + c_2 < 0,$$

then the following are equivalent.

(1) $\exists x \in \mathbb{R}^m$ *such that* $\begin{cases} x^T A_1 x + 2b_1^T x + c_1 < 0 \\ x^T A_2 x + 2b_2^T x + c_2 \leq 0 \end{cases}$,

(2) *there exists no* λ *such that* $\lambda \geq 0$ *and*

$$\begin{pmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{pmatrix} + \lambda \begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} \succeq 0.$$

We can also rewrite this result in a different way, namely

Theorem. *Let* $f, g \colon \mathbb{R}^m \to \mathbb{R}$ *be quadratic functions and suppose that there is an* $\overline{x} \in \mathbb{R}^m$ *such that* $g(\overline{x}) < 0$. *Then the following conditions are equivalent:*

1) *There exists no* $x \in \mathbb{R}^m$ *such that* $f(x) < 0$ *and* $g(x) \leq 0$.

2) *There exists a non negative number* $y \geq 0$ *such that* $f(x) + yg(x) \geq 0$, *for all* $x \in \mathbb{R}^m$.

We see in this case that the S-procedure is an extension of the Farkas theorem which is stated in a same manner except that f and g are convex instead of quadratic functions.

Before giving the proof, we need some more results.

Proposition 4.4. *If* A, $B \in S^m$, *then for all matrices* X *symmetric positive semidefinite, there exists an* $x \in \mathbb{R}^m$ *such that*

$$x^T A x = \textbf{\textit{Tr}}\ (AX), \quad x^T B x = \textbf{\textit{Tr}}\ (BX).$$

The proof of this result is given in (3, Appendix B.3), the book written by Boyd and Vandenberghe.

The next results are given as an example and a exercise in the same book (3, Example 5.14, page 270), where 2. corresponds to Exercise 5.44.

Proposition 4.5.

1. *Let* $Z \in S^k$, F_i,$G \in S^k$, $i = 1, \dots, n$, *then*

$$F(x) = x_1 F_1 + \cdots + x_n F_n + G \prec 0$$

if and only if

$$Z \succeq 0, Z \neq 0, \textbf{\textit{Tr}}\ (GZ) \geq 0, \textbf{\textit{Tr}}\ (F_i Z) = 0, \forall i.$$

2. *Moreover, if*

$$\sum_{i=1}^{n} v_i F_i \succeq 0 \text{ implies } \sum_{i=1}^{n} v_i F_i = 0,$$

Then

$$F(x) = x_1 F_1 + \cdots + x_n F_n + G \preceq 0$$

if and only if

$$Z \succeq 0, \textbf{\textit{Tr}}\ (GZ) > 0, \textbf{\textit{Tr}}\ (F_i Z) = 0, \forall i.$$

Proof of Theorem 4.3. It can easily be shown that (S_1') and (S_2') are weak alternatives. Indeed if we suppose both, the existence of an x in (1) and the existence of a λ in (S_2'), it is obtainned

$$\begin{aligned}
0 &\leq \{\begin{pmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{pmatrix} + \lambda \begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} \text{ is positive definite}\} \\
&\leq \begin{pmatrix} x \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{pmatrix} + \lambda \begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} \right) \begin{pmatrix} x \\ 1 \end{pmatrix} \\
&= \underbrace{x^T A_1 x + 2b_1^T x + c_1}_{<0} + \underbrace{\underbrace{\lambda}_{\geq 0} \underbrace{x^T A_2 x + 2b_2^T x + c_2}_{\leq 0}}_{\leq 0} \\
&< 0,
\end{aligned}$$

which is a contradiction.

Let us now prove the equivalence. We remark that the strict feasibility of $\widehat{x}$ implies that $\begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix}$ has at least one negative eigenvalue. Indeed

$$\begin{pmatrix} \widehat{x} \\ 1 \end{pmatrix}^T \begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} \begin{pmatrix} \widehat{x} \\ 1 \end{pmatrix} = \widehat{x}^T A_2 \widehat{x} + 2b_2^T \widehat{x} + c_2 < 0,$$

and thus $\begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix}$ is not positive semidefinite.

Hence if there exists $\tau \geq 0$ such that $\tau \begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} \succeq 0$, then $\tau = 0$. So by Proposition (4.5), we have the following equivalence.

There exists no λ such that $\lambda \geq 0$ and $\begin{pmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{pmatrix} + \lambda \begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} \succeq 0$

if and only if

$\exists X \succeq 0$ such that $\mathbf{Tr}(X \begin{pmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{pmatrix}) < 0$ and $\mathbf{Tr}(X \begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix}) \leq 0$.

By Proposition (4.4), this is equivalent to

$$\begin{aligned}
\exists \begin{pmatrix} v \\ w \end{pmatrix} \in \mathbb{R}^{m+1} \text{ such that } \mathbf{Tr}(X \begin{pmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{pmatrix}) &= \begin{pmatrix} v \\ w \end{pmatrix}^T \begin{pmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} < 0 \\
\text{and } \mathbf{Tr}(X \begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix}) &= \begin{pmatrix} v \\ w \end{pmatrix}^T \begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \leq 0.
\end{aligned}$$

We now consider the two cases $w = 0$ and $w \neq 0$.

If $w \neq 0$, then the searched vector is $\begin{pmatrix} x \\ 1 \end{pmatrix}$, where $x = v/w$. Indeed, we have

$$x^T A_1 x + 2b_1^T x + c_1 < 0 \text{ and } x^T A_2 x + 2b_2^T x + c_2 \leq 0.$$

If $w = 0$, then $\begin{pmatrix} v \\ w \end{pmatrix}^T \begin{pmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} < 0$ and $\begin{pmatrix} v \\ w \end{pmatrix}^T \begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \leq 0$ reduce to

$$v^T A_1 v < 0 \text{ and } v^T A_2 v \leq 0.$$

Let us consider $x = \widehat{x} + tv$ and calculate

$$\begin{aligned} x^T A_1 x + 2b_1^T x + c_1 &= \widehat{x}^T A_1 \widehat{x} + 2b_1^T \widehat{x} + c_1 + t^2 \underbrace{v^T A_1 v}_{<0} + 2t(A_1 \widehat{x} + b_1)^T v \\ x^T A_2 x + 2b_2^T x + c_2 &= \underbrace{\widehat{x}^T A_2 \widehat{x} + 2b_2^T \widehat{x} + c_2}_{<0} + \underbrace{t^2 v^T A_2 v}_{\leq 0} + 2t(A_2 \widehat{x} + b_2)^T v \\ &< 2t(A_2 \widehat{x} + b_2)^T v. \end{aligned}$$

Hence x is the solution sought when t goes to $\pm\infty$ depending on the sign of $(A_2 \widehat{x} + b_2)^T v$. □

This proof is based on the one given in (3, Appendix B.4).

We will now give alternative proof of Theorem 4.3 inspired by the one given in (17), but we first need an auxiliary result.

Theorem 4.6 (Dines). *If f, $g\colon \mathbb{R}^n \to \mathbb{R}$ are homogeneous quadratic functions, then the set*

$$\mathfrak{M} = \{(f(x), g(x)) \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^2$$

is convex.

Proof. We refer to (17) for the proof of this result. □

Alternative proof of Theorem 4.3.

As we already mentioned the fact that (S_2') implies (S_1') can be checked easily. Let us now assume that there is no $x \in \mathbb{R}^n$ such that $f(x) < 0$ and $g(x) \leq 0$.

Let us first assume that the function f and g are homogeneous, then Dines Theorem, 4.6, assures that the two dimensional image of $\mathbb{R}^n$ under the mapping (f, g) has no common point with the convex cone $C = \{(u_1, u_2) \mid u_1 < 0, u_2 \leq 0\} \subset \mathbb{R}^2$, it is a convex image and therefore the image and C can be separated by a line. In other words, there exist real numbers y_1 and y_2 such that

$$y_1 u_1 + y_2 u_2 \leq 0 \forall (u_1, u_2) \in C \text{ and } y_1 f(x) + y_2 g(x) \geq 0 \forall x \in \mathbb{R}^n.$$

Consider for example $(-3, 0) \in C$, this implies that $y_1 \geq 0$. In a same manner considering the point $(-\varepsilon, -3)$ with ε arbitrarily small, we get that $y_2 \geq 0$. Finally if we consider $\overline{x}$ in the second inequality, we get that $y_1 \neq 0$. Thus $y_1 > 0$ and we can let $y = y_2/y_1$. Hence we have $y \geq 0$ that satisfies $f(x) + yg(x) \geq 0$ for all x.

We now need to prove the same result for any quadratic function f and g. Without loss of generality, we assume that $\overline{x} = 0$. Let us consider explicit expressions of the quadratic functions as follows:

$$f(x) = x^T A_f x + b_f^T x + c_f \text{ and } g(x) = x^T A_g x + b_g^T x + c_g, \text{ with } c_g < 0.$$

Let us now define the homogeneous functions $\tilde{f}, \tilde{g} \colon \mathbb{R}^{n+1} \to \mathbb{R}$ corresponding to f and g

$$\tilde{f}(x,\tau) = x^T A_f x + \tau b_f^T x + \tau^2 c_f, \quad \tilde{g}(x,\tau) = x^T A_g x + \tau b_g^T x + \tau^2 c_g.$$

Suppose there exists $(x, \tau) \in \mathbb{R}^{n+1}$ such that $\tilde{f}(x,\tau) < 0$ and $\tilde{g}(x,\tau) \leq 0$ and let us show that it is not possible, which would imply, since $\tilde{g}(0,1) = g(0) < 0$, that there exists $y \geq 0$ such that $\tilde{f}(x,\tau) + y\tilde{g}(x,\tau) \geq 0$ and the theorem is proven by choosing $\tau = 1$.

Consider first that $\tau \neq 0$, then $f(x/\tau) = \tilde{f}(x,\tau)/\tau^2 < 0$ and $g(x/\tau) = \tilde{g}(x)/\tau^2 \leq 0$, which contradicts our first hypothesis.

Now if $\tau = 0$, then $x^T A_f x < 0$ and $x^T A_g x \leq 0$ and thus $\underbrace{(x\lambda)^T A_f (x\lambda)}_{<0} + \lambda b_f^T x + c_f < 0$ for $\mid \lambda \mid$ large enough and $\underbrace{(x\lambda)^T A_g (x\lambda)}_{\leq 0} + \lambda b_g^T x + \underbrace{c_g}_{<0} < 0$ if λ has the proper sign, which is also a contradiction to the first assumption.

$\square$

We will now give an example of use of the S-procedure to control problems. The example we consider in this paper is given in (19) or in (8).

Example 4.1. *Consider the following system dynamic*

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx, \quad x(0) = x_0. \end{cases} \tag{4.0.1}$$

We want to find a necessary and sufficient condition for (4.0.1) to be quadratically stable, where u and y have to satisfy the sector constraint

$$\sigma(y,u) = (\beta y - u)^T (u - \alpha y) \geq 0,$$

with $\alpha < \beta \in \mathbb{R}$. That is, we search a symmetric matrix P such that

$$\begin{aligned} &V(x) = x^T P x \textit{ and} \\ &\dot{V}(x) = 2x^T P(Ax + Bu) < 0, \ \forall (x,u) \neq 0 \textit{ s.t. } \sigma(x,u) \geq 0. \end{aligned} \tag{4.0.2}$$

Let us now define

$$\begin{aligned} \sigma_0(x,u) &= \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} A^T P + PA & PB \\ B^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\ \sigma_1(x,u) = 2\sigma(Cx,u) &= \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} -2\beta\alpha C^T C & (\beta+\alpha)C^T \\ (\beta+\alpha)C & -2 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \end{aligned}$$

then we can rewrite condition (4.0.2) in terms of σ_0 *and* σ_1*. Thus the condition on* V *that needs to be checked is*

$$\sigma_0(x,u) < 0, \forall (x,u) \neq 0 \ s.t. \ \sigma_1(x,u) \geq 0. \tag{4.0.3}$$

As $\alpha < \beta$*, the function* σ_1 *is regular and hence, we can apply the S-procedure. According to Theorem 4.3, the S-procedure is lossless and therefore we can check if there exists* $\tau \geq 0$ *such that*

$$\begin{pmatrix} A^TP + PA & PB \\ B^TP & 0 \end{pmatrix} + \tau \begin{pmatrix} -2\beta\alpha C^TC & (\beta+\alpha)C^T \\ (\beta+\alpha)C & -2 \end{pmatrix} \prec 0, \tag{4.0.4}$$

for all $(x,u) \neq 0$.

Since $\begin{pmatrix} A^TP + PA & PB \\ B^TP & 0 \end{pmatrix}$ *is not necessarily negative definite, we have that* $\tau > 0$*. Therefore, we can divide everything by* τ *and rename* P/τ *by* P*.*

Finally the system is quadratically stable if and only if

$$\begin{pmatrix} A^TP + PA - 2\beta\alpha C^TC & PB + (\beta+\alpha)C^T \\ B^TP + (\beta+\alpha)C & -2 \end{pmatrix} \prec 0,$$

which is a well-known linear matrix inequality. This example is known as the circle criterion.

Moreover since it holds that $(A^TP + PA - 2\alpha\beta C^TC)^T = PA + A^TP - 2\alpha\beta C^TC$*,* $(B^TP + (\beta+\alpha)C)^T = PB + (\beta+\alpha)C^T$ *and* $(-2)^T = -2 < 0$*, we can apply the* SCHUR*'s lemma to this matrix inequality, thus it is equivalent to the following inequality:*

$$\begin{aligned} &(A^TP + PA + 2\alpha\beta C^TC) \\ &\qquad\qquad - (PB - (\beta+\alpha)C^T)(1/2)(B^TP - (\beta+\alpha)C) \prec 0 \\ &A^TP + PA - PB(1/2)B^TP \\ &\qquad\qquad + 2\alpha\beta C^TC + 1/2(PBC + C^TB^TP)(\beta+\alpha) \prec 0. \end{aligned}$$

We recognize that this inequality is a RICCATI *inequality.*

By considering the case where $C = 0$*, that is without any* y *nor any constraint on* u*, we get the inequality* $A^TP + PA - PB(-1/2)B^TP \prec 0 \prec I$*, that can be rewritten as*

$$A^TP + PA - I - PB(-1/2)B^TP \preceq 0. \tag{4.0.5}$$

This last equation (4.0.5), seen as an equality, corresponds to the usual maximization deterministic problem, see (3.1.1), with $Q = I$ *and* $R = 2$*.*

Chapter 5

Conclusion

Finally, in this work we were able to first understand the difference between duality results in semidefinite programming and linear programming, then give some methods to solve semidefinite programming problem, give a way through the S-procedure, to check stability of the system dynamics in control problems by solving a semidefinite programming condition and finally apply semidefinite programming theory to control theory, in the particular case of stochastic linear quadratic control problems over infinite time horizon.

When speaking about control theory, we could verify some stability conditions through semidefinite programming condition in Chapter 4. But also, in Chapter 3, we could derive the Riccati equation for both nature of the control, replace the Riccati equation by a primal and a dual semidefinite programming problem.

The generalized Riccati equation was also introduced with use of the pseudo inverse of a matrix and stability results for the control depending on both, the semidefinite programming problems and the generalized Riccati equation could be proven.

Then we managed to show some optimality results linking the semidefinte programming primal-dual problems, the generalized Riccati equation and the control problem.

Thus we came to an algorithm to solve stochastic linear quadratic control problems over infinite time horizon through semidefinite programming or solving some linear matrix (in)equalities for the obtainning a solution to the generalized Riccati equation. We ended by some examples of control problems, which the algorithm was applied to.

Our paper is mainly a review of the article (28) by Yao, Zang and Zhou. We mention here that there exists other versions (24; 26) of the article (28) by Yao, Zang and Zhou, but that we based our sayings on (28).

Therefore, we considered only the case of stochastic linear quadratic control problem over infinite time horizon and hence autonomous systems.

We could have for instance considered stochastic linear quadratic control problems with explicit time dependence, so over finite time horizon $[0, T]$, where T is to be optimized as in the article (1) by Rami, Moore and Zhou.

Another improvement of this article could have been to treat examples of higher dimension as Yao, Zang and Zhou are doing in (27; 25).

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