REDUCED FORMULATION OF A STEADY-FLUID STRUCTURE INTERACTION PROBLEM WITH PARAMETRIC COUPLING

TONI LASSILA AND GIANLUIGI ROZZA

ABSTRACT. We propose a two-fold approach to model reduction of fluid-structure interaction. The state equations for the fluid are solved with reduced basis methods. These are model reduction methods for parametric partial differential equations using well-chosen snapshot solutions in order to build a set of global basis functions. The other reduction is in terms of the geometric complexity of the moving fluid-structure interface. We use free-form deformations to parameterize the perturbation of the flow channel at rest configuration. As a computational example we consider a steady fluid-structure interaction problem: an incompressible Stokes flow in a channel that has a flexible wall.

1. INTRODUCTION

Many problems in fields such as aerodynamics and biomechanics can be expressed as fluid-structure interaction (FSI) problems. The mathematical modelling of such coupled problems consists of four main parts: solution of the fluid equations given the current fluid geometry, solution of the structural displacements given the normal stresses exerted by the fluid, fulfillment of coupling constraints to achieve force balance across the interface, and transport of the fluid-structure interface. Even when both the fluid and structure equations are independently linear, the geometric variability of the fluid-structure interface yields a significant nonlinearity in the coupled system. Solution methods for fluid-structure interaction problems are iterative in nature and involve the repeated solution of the fluid and structure equations in many different configurations.

Fluid-structure interaction problems arise in modelling of the arterial deformations in the human cardiovascular system. A comprehensive treatment of FSI strategies in cardiovascular modelling can be found in [13, 15]. Blood flow is pulsatile and the displacements of the arterial walls are relatively large. This leads to stability considerations that necessitate the use of implicit solver strategies of the coupled FS problem, again increasing computational complexity.

Model Order Reduction (MOR) is a cross-disciplinary field that strives to systematically reduce the dynamics of a complicated ordinary or partial differential equation model to a simpler and computationally more tractable one. Model reduction techniques have also been proposed for nonlinear systems. One particularly common approach is the Proper Orthogonal Decomposition (POD) technique that aims at decomposing the dynamics of a time-dependent system into fundamental modes, and then choosing only very few of the most important modes to represent the entire dynamics of the system.

A modal reduction of the structural equations was adopted in [11] to reduce the complexity of an FSI problem involving pulsatile flow in a channel, but no attempt to reduce the complexity of solving the fluid equations was made. In another work by the same author [10] the steady problem was parameterized using shape functions for the boundary trace of the pressure obtained from an assumption of near-Poiseuille flow profile. The strong coupling of normal stresses on the interface was treated in the reduced space obtained by considering only the leading structural eigenmodes, and solved as a least-squares optimization problem.

Another approach consists of the Reduced Basis (RB) methods, which is generally used on parametrized partial (and ordinary) differential equations. RB methods are based on a greedy sampling algorithm that chooses snapshot solutions of the parametric PDE at different parameter values for constructing a global approximation basis. For an introduction, see [14] and [20]. The reduction of steady incompressible Stokes flow (without FSI) in an arterial bypass configuration by parameterization with piecewise affine maps was considered in [21]. The same problem parameterized with nonaffine maps was considered in [19] for Stokes equations and in [16] for Navier-Stokes equations. We also mention the reduced basis element method proposed for Stokes flows in [8] that was derived to address complex flow networks with multiple
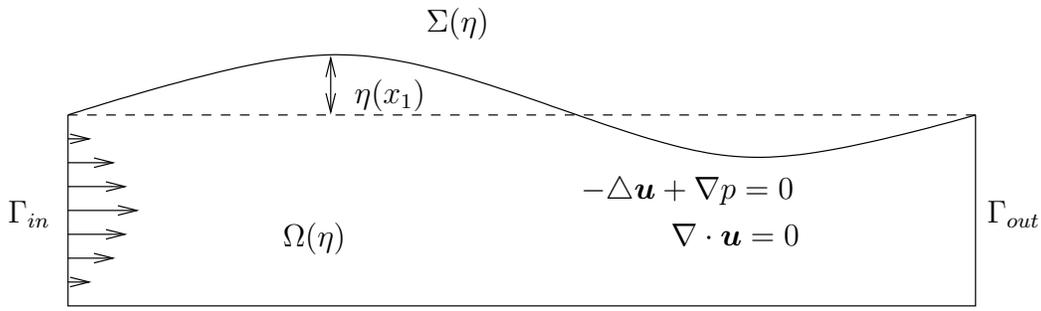
FIGURE 1. Stokes flow in a channel with a flexible wall section

branches and junctions. A posteriori error estimation for Navier-Stokes equations solved by reduced basis methods is considered in [2, 23]. We use reduced basis methods to approximate the solution of the Stokes equations in a parametric flow geometry.

2. PROBLEM OF STEADY STOKES FLOW IN A FLEXIBLE CHANNEL

We denote by $H^m(X)$ the usual Sobolev space of real-valued functions on X with mth weak derivatives being square-integrable. Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega_0$ that represents the fluid domain at rest configuration (without deformations induced by the fluid flow). We denote the flexible part of the boundary as $\Sigma_0 \subset \partial\Omega_0$. We assume the geometry displayed in Fig. 1 – a straight 2-d tube, where the displacement of the upper wall is described with a function $\eta \in H_0^1(\Sigma_0) := \{\phi \in H^1(\Sigma_0) : \phi(a) = \phi(b) = 0\}$. The tube is at "rest" configuration when $\eta \equiv 0$. We denote by $\Omega(\eta)$ the deformed domain. In the deformed domain we have the incompressible steady Stokes problem to find $\boldsymbol{u} \in [H^1(\Omega(\eta))]^2$ and $p \in L_0^2(\Omega(\eta))$ s.t.

$$
(1) \quad
\begin{cases}
\displaystyle\int_{\Omega(\eta)} [\nu\nabla\boldsymbol{u} \cdot \nabla\boldsymbol{v} - p\nabla \cdot \boldsymbol{v}] \, d\Omega = \int_{\Omega(\eta)} \boldsymbol{f}^F \cdot \boldsymbol{v}, & \text{for all } \boldsymbol{v} \in [H^1(\Omega(\eta))]^2 \\[2ex]
\displaystyle\int_{\Omega(\eta)} q\nabla \cdot \boldsymbol{u} \, d\Omega = 0\,, & \text{for all } q \in L_0^2(\Omega(\eta)) \\[2ex]
\boldsymbol{u} = \boldsymbol{u}_0 \text{ on } \partial\Omega(\eta) \setminus \Sigma(\eta), \quad \boldsymbol{u} = \boldsymbol{0} & \text{on } \Sigma(\eta)
\end{cases}
$$

where ν is the viscosity, and $\boldsymbol{f}^F$ is the volume force acting on the fluid. The Dirichlet data are assumed to satisfy a conservation principle

$$
(2) \quad \int_{\partial\Omega(\eta)} \boldsymbol{u}_0 \cdot \boldsymbol{n} \, d\Gamma = 0.
$$

The fluid equations are then to be coupled together with the structural equations given the coupling condition that the displacement of the structure is the same as would be obtained from the structural equations, where the RHS is the stress exerted by the fluid on the interface. The structural equations can be chosen in several different ways (in [5] and [11] different fourth order models are considered). We choose, after a suitable assumption of small structural displacement, to model the displacement of the wall using a 1-d elliptic equation ("elastic membrane") to find $\eta \in H_0^1(\Sigma_0)$ s.t.

$$
(3) \quad \int_{\Sigma_0} K(\boldsymbol{x})\eta'\phi' \, d\Gamma = \int_{\Sigma_0} \tau(\boldsymbol{u}, p)\phi \, d\Gamma, \quad \forall\, \phi \in H_0^1(\Sigma_0)
$$

where we define the space $H_0^1(\Sigma_0) := \{\phi \in H^1(\Sigma_0) : \phi(a) = \phi(b) = 0\}$, and $\tau(\boldsymbol{u}, p)$ is the normal traction on the interface imposed by the fluid

$$
(4) \quad \tau(\boldsymbol{u}, p) = \left[p\boldsymbol{n} - \nu\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right) \cdot \boldsymbol{n} \right]^T \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

obtained from the solution $(\boldsymbol{u}, p)$ of the fluid equation (1), and $K(\boldsymbol{x}) \geq K_0 > 0$ is the spring constant of the wall to displacements in the normal direction.

In operator form, the coupled fluid-structure problem can be considered to be of the form

$$
(5) \qquad
\begin{bmatrix} -\nu\triangle & \nabla & 0 \\ \operatorname{div} & 0 & 0 \\ 0 & 0 & K\partial_{xx} \end{bmatrix}
\begin{bmatrix} \boldsymbol{u} \\ p \\ \eta \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ \tau(\boldsymbol{u},p) \end{bmatrix}
=
\begin{bmatrix} \boldsymbol{f}^F \\ 0 \\ 0 \end{bmatrix}.
$$

Because the fluid domain depends on the domain defined by the function η, this is in fact a nonlinear free-boundary problem.

3. PARAMETERIZATION WITH FREE-FORM DEFORMATIONS

To reduce the complexity of the free-boundary problem (5) we introduce a parameterization of the fluid domain, $\Omega(\boldsymbol{\mu})$, where the parameters $\boldsymbol{\mu} \in \mathcal{D}$ belong to some low-dimensional parameter space. One option to construct $\Omega(\boldsymbol{\mu})$ would be to use a family of parametric curves, such as B-splines, to directly parameterize the displacement function η. We prefer to use *free-form deformations* [22].

Free-form deformations are a technique of obtaining small, parametric deformations of an arbitrary reference domain $\Omega_0 \subset \mathbb{R}^d$ for $d = 2,3$. First we map the reference domain to a subset of the unit square with a continuous invertible map $\Psi : \Omega_0 \to (0,1) \times (0,1)$ and then overlay on $\Psi(\Omega_0)$ a regular grid of control points $\boldsymbol{P}^0_{\ell,m}$, $\ell = 0,\ldots,L$ and $m = 0,\ldots,M$, so that $\boldsymbol{P}^0_{\ell,m} = \begin{bmatrix} \ell/L & m/M \end{bmatrix}^T$. The parameterization is obtained by allowing a subset of the control points to move and then using a spline basis to construct a smooth deformation map as a function of the positions of the control points. The perturbed control points are given by a set of $(L+1)(M+1)$ parameter vectors $\boldsymbol{\mu}_{\ell,m}$. For each $\boldsymbol{\mu}$, a parametric domain map $T : \Omega_0 \times \mathcal{D} \to \Omega(\boldsymbol{\mu})$ is defined as

$$
(6) \qquad
T(\boldsymbol{x};\boldsymbol{\mu}) = \Psi^{-1}\left(\sum_{\ell=0}^{L} \sum_{m=0}^{M} b_{\ell,m}^{L,M}(\Psi(\boldsymbol{x}))\left(\boldsymbol{P}_{\ell,m} + \boldsymbol{\mu}_{\ell,m} \right) \right),
$$

where $b_{\ell,m}^{L,M}$ are suitable polynomial basis functions. Here we take

$$
b_{\ell,m}^{L,M}(s,t) = \binom{L}{\ell}\binom{M}{m}(1-s)^{(L-\ell)}s^{\ell}(1-t)^{(M-m)}t^{m},
$$

that is the tensor products of the 1-d Bernstein basis polynomials

$$
b_{\ell}^{L}(s) = \binom{L}{\ell}(1-s)^{(L-\ell)}s^{\ell}, \quad b_{m}^{M}(t) = \binom{M}{m}(1-t)^{(M-m)}t^{m}
$$

defined on the unit square with local variables $(s,t) \in (0,1) \times (0,1)$. It is typical that only some of the control points are allowed to move freely, we do not use the full parameterization but only a few of the most relevant shape parameters. To parameterize the flow channel we use a 10×2 grid of control points. Six control points in the top row are moving in the x_2-direction, resulting in a smooth parameterized deformation of the channel with 6 real parameters. The result when applied on a sample computational mesh is shown in Fig. 2.

After the parametric domain $\Omega(\boldsymbol{\mu})$ is obtained as the image of a fixed reference domain Ω_0 under the map $T : \Omega_0 \times \mathcal{D} \to \Omega(\boldsymbol{\mu})$, which is assumed to be differentiable and invertible, we proceed to transform the PDE on a parametric domain to a parametric PDE on the fixed reference domain. Denoting by $J_{T_\eta} := \nabla_{\boldsymbol{x}} T_\eta$ the Jacobian matrix of T_η we define the transformation tensors for the viscous term

$$
(7) \qquad \nu_{T_\eta} := J_{T_\eta}^{-T} J_{T_\eta}^{-1} \det(J_{T_\eta})
$$

and the pressure-divergence term

$$
(8) \qquad \chi_{T_\eta} := J_{T_\eta}^{-1} \det(J_{T_\eta})
$$

respectively. The inhomogeneous Dirichlet conditions are handled by lifting to a space of proper boundary conditions the solution of the homogeneous Stokes problem on the reference domain to find $\boldsymbol{u}(\eta) \in$
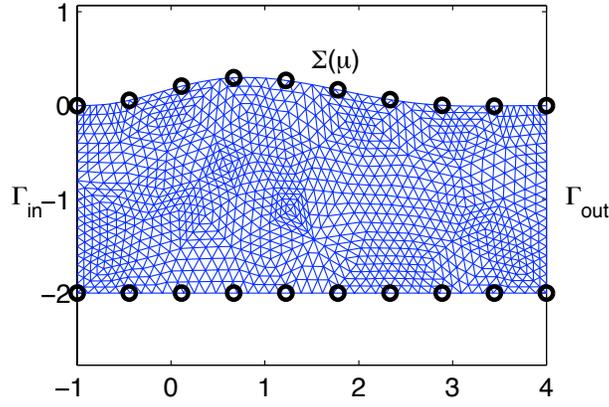
FIGURE 2. Free-form deformation of the flexible channel with six parameters.

$[H_0^1(\Omega_0)]^2$ and $p(\eta) \in L_0^2(\Omega_0)$ s.t.

$$(9) \quad \begin{cases} \displaystyle\int_{\Omega_0} \left(\nu \frac{\partial u_k}{\partial x_i} [\nu_{T_\eta}]_{i,j} \frac{\partial v_k}{\partial x_j} + p\,[\chi_{T_\eta}]_{k,j} \frac{\partial v_k}{\partial x_j} \right) d\Omega_0 = \int_{\Omega_0} \det(J_{T_\eta})[f^F + f_{\text{lift}}]_k \, d\Omega_0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \boldsymbol{v} \in H_0^1(\Omega_0) \times H_0^1(\Omega_0) \\ \displaystyle\int_{\Omega_0} q\,[\chi_{T_\eta}]_{k,j} \frac{\partial u_k}{\partial x_j} \, d\Omega_0 = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, q \in L_0^2(\Omega_0) \end{cases}$$

with summation understood over the indices $i = 1, 2$ and $j = 1, 2$. In a similar way the structural equations are transferred from $\Sigma(\eta)$ to Σ_0.

4. PARAMETRIC FLUID-STRUCTURE COUPLING ALGORITHM

We can alternatively formulate the fluid-structure problem (5) as a calculus of variations problem to find $\eta \in H_0^1(\Sigma_0)$ the minimizer of

$$(10) \qquad\qquad \min_\eta \quad \tfrac{1}{2} \int_{\Sigma_0} |K \partial_{xx} \eta + \tau(\boldsymbol{u}, p)|^2 \, d\Gamma,$$

$$\text{s.t.} \quad \begin{cases} -\nu \triangle \boldsymbol{u} + \nabla p = \boldsymbol{f}^F & \text{in } \Omega(\eta) \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega(\eta) \end{cases}$$

If a solution of (5) exists then it is also a minimizer of (10). Problem (10) in the parametric form given by (9) is still valid: find $\boldsymbol{\mu} \in \mathcal{D}$ s.t.

$$(11) \qquad\qquad \min_{\boldsymbol{\mu}} \quad \tfrac{1}{2} \int_{\Sigma_0} |K \partial_{xx} \eta + \tau(\boldsymbol{u}, p)|^2 \, d\Gamma,$$

$$\text{s.t.} \quad \begin{cases} -\nu \triangle \boldsymbol{u} + \nabla p = \boldsymbol{f}^F & \text{in } \Omega(\boldsymbol{\mu}) \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega(\boldsymbol{\mu}) \end{cases}$$

but this time we expect that the optimal value of the functional is $J^* > 0$ and the coupling of stresses on the interface between fluid and structure is only achieved in a least-squares sense. The "goodness of fit" depends on the dimension of the parameter space $\mathcal{D}$. We call this the *strong parametric coupling* approach.

It should be noted that we perform a reduction by parameterizing the displacement η of the fluid-structure interface. In [10] the traction τ (which corresponds to the boundary trace of the pressure in

this case) was instead parameterized. This alternative approach allowed the author to solve directly a parametric minimization problem of the form (11). Instead, we must consider the minimization problem

$$(12) \qquad \min_{\boldsymbol{\mu}} \quad \tfrac{1}{2} \int_{\Sigma_0} \left[|\eta(\boldsymbol{\mu}) - \hat{\eta}|^2 + |\partial_x \eta(\boldsymbol{\mu}) - \partial_x \hat{\eta}|^2 \right] \, d\Gamma,$$

$$\text{s.t.} \quad \begin{cases} -\nu \triangle \boldsymbol{u} + \nabla p = \boldsymbol{f}^F & \text{in } \Omega(\boldsymbol{\mu}) \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega(\boldsymbol{\mu}) \\ K \partial_{xx} \hat{\eta} + \tau(\boldsymbol{u}, p) = 0 & \text{on } \Sigma(\boldsymbol{\mu}) \end{cases} .$$

The reason for this is that, in order to reduce the complexity of the fluid equations, we must be able to transfer the parametric dependence to the Stokes equations. This can be done more easily by parameterizing the displacement rather than the boundary trace of the pressure.

A computational algorithm for the solution of (12) is detailed in Algorithm 1. This is a fixed point

---

Algorithm 1 Strong parametric coupling of fluid and structure

---

Require: initial guess $\boldsymbol{\mu}^0$

1: Let $k = 0$.

2: **repeat**

3:     Solve the discretized fluid equations for $(\boldsymbol{u}^k, p^k)$ in $\Omega(\boldsymbol{\mu}^k)$.

4:     Compute assumed traction $\hat{\tau} = p^k|_{\Sigma(\boldsymbol{\mu})} - \nu \left[\nabla \boldsymbol{u}^k + [\nabla \boldsymbol{u}^k]^T \right]$ from $(\boldsymbol{u}^k, p^k)$.

5:     Solve the minimization problem

$$\boldsymbol{\mu}^{k+1} := \operatorname*{argmin}_{\boldsymbol{\mu} \in \mathcal{D}} \quad \tfrac{1}{2} \int_{\Sigma_0} \left\{ |\eta(\boldsymbol{\mu}) - \hat{\eta}|^2 + |\partial_x \eta(\boldsymbol{\mu}) - \partial_x \hat{\eta}|^2 \right\} \, d\Gamma$$

    to obtain a new configuration parameter, where $\eta(\boldsymbol{\mu})$ is the displacement of the interface given by the parameterization of the geometry and the *assumed displacement* $\hat{\eta} \in H_0^1(\Sigma_0)$ is the solution of

$$\int_{\Sigma_0} K \hat{\eta}' \phi' \, d\Gamma = \int_{\Sigma_0} \hat{\tau} \phi \, d\Gamma \quad \forall \phi \in H_0^1(\Sigma_0)$$

6:     Set $k := k + 1$.

7: **until** stopping criteria $|\boldsymbol{\mu}^{k+1} - \boldsymbol{\mu}^k| < \varepsilon$ is met

---

algorithm in the parameter space. Existence of a fixed point and convergence of the algorithm will be addressed in a future work. The algorithm's computationally most intensive part is the solution of the parameterized fluid equations. We next discuss the reduced basis method for approximation of the fluid solution $(\boldsymbol{u}^k, p^k)$.

5. REDUCED BASIS APPROXIMATION OF THE STOKES PROBLEM

The computational efficiency of solving the coupling problem (12) hinges on the efficient solution of the fluid equations on the parametric domain $\Omega(\boldsymbol{\mu})$. We present briefly the general approach to reduced basis approximations for the steady Stokes problem. For more details see [19] and [21].

We denote by $\mathcal{Y} := H^1(\Omega_0) \times H^1(\Omega_0)$ the velocity space, $\mathcal{Q} := L^2(\Omega_0)$ the pressure space, and the continuous parametric bilinear forms $\mathcal{A} : \mathcal{Y} \times \mathcal{Y} \times \mathcal{D} \to \mathbb{R}$ and $\mathcal{B} : \mathcal{Q} \times \mathcal{Y} \times \mathcal{D} \to \mathbb{R}$. The weak form (9) of the parametric Stokes equations for velocity and pressure can be written: given any $\boldsymbol{\mu} \in \mathcal{D}$, find $\boldsymbol{u}(\boldsymbol{\mu}) \in \mathcal{Y}$ and $p(\boldsymbol{\mu}) \in \mathcal{Q}$ s.t.

$$(13) \qquad \begin{aligned} \mathcal{A}(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) + \mathcal{B}(p, \boldsymbol{v}; \boldsymbol{\mu}) &= \langle \boldsymbol{F}(\boldsymbol{\mu}), \boldsymbol{v} \rangle_{\mathcal{Y}} & \text{for all } \boldsymbol{v} \in \mathcal{Y} \\ \mathcal{B}(q, \boldsymbol{u}(\boldsymbol{\mu})) &= \langle G(\boldsymbol{\mu}), q \rangle_{\mathcal{Q}} & \text{for all } q \in \mathcal{Q}, \end{aligned}$$

where $\boldsymbol{F}(\boldsymbol{\mu}) \in \mathcal{Y}' \times \mathcal{D}$ and $G(\boldsymbol{\mu}) \in \mathcal{Q}' \times \mathcal{D}$ are continuous parametric linear forms. The inhomogeneous Dirichlet condition is treated by introducing an extension of the boundary data $\tilde{\boldsymbol{u}}_0 \in H^1(\Omega_0)$ and looking for a solution $\tilde{\boldsymbol{u}} \in H_0^1(\Omega_0) \times H_0^1(\Omega_0) \times \mathcal{D}$ to (13) with a modified right-hand side $\langle \tilde{\boldsymbol{F}}(\boldsymbol{\mu}), \boldsymbol{v} \rangle_{\mathcal{Y}} := \langle \boldsymbol{F}(\boldsymbol{\mu}), \boldsymbol{v} \rangle_{\mathcal{Y}} - \langle \mathcal{A}(\boldsymbol{\mu}) \tilde{\boldsymbol{u}}_0, \boldsymbol{v} \rangle_{\mathcal{Y}}$. The solution is then recovered as $\boldsymbol{u}(\boldsymbol{\mu}) = \tilde{\boldsymbol{u}}(\boldsymbol{\mu}) + \mathcal{A}(\boldsymbol{\mu}) \tilde{\boldsymbol{u}}_0$.

Define $\alpha(\boldsymbol{\mu})$ the parameter-dependent coercivity constant of the problem as

$$(14) \qquad \alpha(\boldsymbol{\mu}) := \inf_{\boldsymbol{v} \in \mathcal{Y}} \frac{\mathcal{A}(\boldsymbol{v}, \boldsymbol{v}; \boldsymbol{\mu})}{||\boldsymbol{v}||_{\mathcal{Y}}^2},$$

and $\beta(\boldsymbol{\mu})$ the parameter-dependent inf-sup constant of the problem as

$$(15) \qquad \beta(\boldsymbol{\mu}) := \inf_{q \in \mathcal{Q}} \sup_{\boldsymbol{v} \in \mathcal{Y}} \frac{\mathcal{B}(q, \boldsymbol{v}; \boldsymbol{\mu})}{||q||_{\mathcal{Q}}||\boldsymbol{v}||_{\mathcal{Y}}}.$$

A necessary condition known as the Ladyzhenskaya-Babuška-Brezzi condition for problem (13) to have a unique solution is that $\alpha(\boldsymbol{\mu}) \geq \alpha_0 > 0$ and $\beta(\boldsymbol{\mu}) \geq \beta_0 > 0$. We assume in what follows that the coercivity constant is always positive and concentrate on the inf-sup constant.

From here on we forego any interest in the infinite-dimensional problem and assume instead that both $\mathcal{Y}$ and $\mathcal{Q}$ are replaced by strictly finite-dimensional spaces $\mathcal{Y}_h$ and $\mathcal{Q}_h$. More concretely, we consider only the finite-dimensional problem obtained by discretization of the partial differential equations using the finite element method. The discretized model problem

$$(16) \qquad \begin{aligned} \mathcal{A}(\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) + \mathcal{B}(p_h, \boldsymbol{v}_h; \boldsymbol{\mu}) &= \langle \boldsymbol{F}_h(\boldsymbol{\mu}), \boldsymbol{v}_h \rangle_{\mathcal{Y}_h} \qquad \text{for all } \boldsymbol{v}_h \in \mathcal{Y}_h \\ \mathcal{B}(q_h, \boldsymbol{u}_h; \boldsymbol{\mu}) &= \langle G_h(\boldsymbol{\mu}), q_h \rangle_{\mathcal{Q}_h} \qquad \text{for all } q_h \in \mathcal{Q}_h, \end{aligned}$$

involves a similar inf-sup condition to guarantee the existence of a unique velocity-pressure pair satisfying (16):

$$(17) \qquad \beta_h(\boldsymbol{\mu}) = \inf_{q_h \in \mathcal{Q}_h} \sup_{\boldsymbol{v}_h \in \mathcal{Y}_h} \frac{\mathcal{B}(q_h, \boldsymbol{v}_h; \boldsymbol{\mu})}{||q_h||_{\mathcal{Q}}||\boldsymbol{v}_h||_{\mathcal{Y}}} \geq \beta_0 > 0.$$

In the case of the steady Stokes problem we employ the $\mathbb{P}^2 - \mathbb{P}^1$ Taylor-Hood elements [4, 17] that satisfy the condition (17) without the necessity of adding extra stabilizing terms.

The reduced basis method seeks an approximation of the finite element solution to (16). The first step is to construct a set of global basis functions. Let $\boldsymbol{\mu}^1, \ldots, \boldsymbol{\mu}^N$ be a set of snapshot parameter values chosen according to some rule. We denote by $\boldsymbol{u}(\boldsymbol{\mu}^n)$ and $p(\boldsymbol{\mu}^n)$ the corresponding snapshot solutions for the velocity and pressure. The snapshot solutions are obtained as solutions of a stable finite element formulation of (16). The reduced basis velocity space is then defined as

$$(18) \qquad \text{span}(u(\boldsymbol{\mu}^1), \ldots, u(\boldsymbol{\mu}^N)) =: \mathcal{Y}_N \subset \mathcal{Y}_h,$$

and the reduced basis pressure space as

$$(19) \qquad \text{span}(p(\boldsymbol{\mu}^1), \ldots, p(\boldsymbol{\mu}^N)) =: \mathcal{Q}_N \subset \mathcal{Q}_h.$$

We define the *reduced basis Galerkin* problem: for any $\boldsymbol{\mu} \in \mathcal{D}$ find $u_N \in \mathcal{Y}_N$ and $p_N \in \mathcal{Q}_N$ such that

$$(20) \qquad \begin{aligned} \mathcal{A}(u_N(\boldsymbol{\mu}), v; \boldsymbol{\mu}) + \mathcal{B}(p_N(\boldsymbol{\mu}), v; \boldsymbol{\mu}) &= \langle \boldsymbol{F}_h(\boldsymbol{\mu}), v \rangle \qquad \text{for all } v \in \mathcal{Y}_N \\ \mathcal{B}(q, u_N(\boldsymbol{\mu}); \boldsymbol{\mu}) &= \langle G_h(\boldsymbol{\mu}), q \rangle \qquad \text{for all } q \in \mathcal{Q}_N. \end{aligned}$$

Then $(\boldsymbol{u}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu}))$ is a reduced basis approximation for $(\boldsymbol{u}_h(\boldsymbol{\mu}), p_h(\boldsymbol{\mu}))$ the solution of (16). To represent and solve these equations in matrix form, an orthonormal basis is constructed for $\mathcal{Y}_N$ and $\mathcal{Q}_N$ to guarantee algebraic stability as $N \to \infty$ [21].

Provided that the dimension N is chosen to be much smaller than the dimension of the finite element spaces, this system is inexpensive to solve. If the dependence of the bilinear forms on the parameters is smooth and the snapshot parameter values $\boldsymbol{\mu}_n$ are properly chosen, we find that choosing a moderately small N permits rapid convergence towards the finite element solution, provided that . For details on how to choose the snapshots with the help of a greedy algorithm that controls the residual a posteriori approximation error, we refer to existing works on reduced basis methods [6, 14, 20].

A key issue in the reduced basis approximation of the Stokes problem is approximation stability. We remark that use of inf-sup-stable finite elements for the snapshot computation does not necessarily result in a stable reduced basis approximation when the basis functions are orthonormalized, as was observed in [21]. On the other hand, the orthonormalization is vital for the algebraic stability of the reduced basis approximation. Thus there is a need to extend the concept of inf-sup constants into the reduced basis formulation. This is typically done by defining the so-called supremizers in velocity space and enriching

the reduced basis velocity space $\mathcal{Y}_N$ accordingly. Following [21] we define the inner supremizer operator $T^{\boldsymbol{\mu}} : \mathcal{Q}_h \to \mathcal{Y}_h$ as the solution of the elliptic problem

$$(21) \qquad \langle T^{\boldsymbol{\mu}} q, \boldsymbol{v} \rangle_{\mathcal{Y}} = \mathcal{B}(q, \boldsymbol{v}; \boldsymbol{\mu}) \quad \text{for all } \boldsymbol{v}_h \in \mathcal{Y}_h.$$

After the pressure approximation space $\mathcal{Q}_N$ and velocity approximation space $\mathcal{Y}_N$ have been constructed we add all the velocity supremizers, i.e. velocity fields of the form $T^{\boldsymbol{\mu}}(p(\boldsymbol{\mu}^n))$, to the velocity space:

$$(22) \qquad \tilde{\mathcal{Y}}_N := \mathcal{Y}_N \cup \text{span} \left[T^{\boldsymbol{\mu}}(p(\boldsymbol{\mu}^1)), \dots, T^{\boldsymbol{\mu}}(p(\boldsymbol{\mu}^N)) \right],$$

where $\tilde{\mathcal{Y}}_N$ is a stabilized reduced velocity space. Then

$$(23) \qquad \tilde{\beta}_N(\boldsymbol{\mu}) := \inf_{q \in \mathcal{Q}_N} \sup_{\boldsymbol{v} \in \tilde{\mathcal{Y}}_N} \frac{\mathcal{B}(q, \boldsymbol{v}; \boldsymbol{\mu})}{||q||_{\mathcal{Q}_N} ||\boldsymbol{v}||_{\tilde{\mathcal{Y}}_N}} \geq \beta_h(\boldsymbol{\mu}) \geq \beta_0 > 0.$$

As a consequence of this approach to stabilization the dimension of the velocity approximation space grows to $2N$, where N is the original dimension of velocity and pressure approximation spaces respectively.

The computational benefits of the reduced basis method rely on the assumption that the parametric bilinear forms are affinely parameterized:

$$(24) \qquad \mathcal{A}(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) = \sum_{m=1}^{M_a} \Theta_a^m(\boldsymbol{\mu}) \mathcal{A}^m(\boldsymbol{u}, \boldsymbol{v}), \quad \mathcal{B}(p, \boldsymbol{v}; \boldsymbol{\mu}) = \sum_{m=1}^{M_b} \Theta_b^m(\boldsymbol{\mu}) \mathcal{B}^m(p, \boldsymbol{v})$$

for some parameter-independent continuous bilinear forms $\mathcal{A}^m$ and $\mathcal{B}^m$, and parametric coefficient functions Θ_a^m and Θ_b^m that can be readily evaluated. Then the finite element equations (16) split into

$$(25) \qquad \sum_{m=1}^{M_a} \Theta_a^m(\boldsymbol{\mu}) \mathcal{A}^m(\boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{v}) + \sum_{m=1}^{M_b} \Theta_b^m(\boldsymbol{\mu}) \mathcal{B}^m(p_N(\boldsymbol{\mu}), \boldsymbol{v}) = \langle \boldsymbol{F}_h(\boldsymbol{\mu}), \boldsymbol{v} \rangle \qquad \text{for all } \boldsymbol{v} \in \mathcal{Y}_h$$

$$\sum_{m=1}^{M_b} \Theta_b^m(\boldsymbol{\mu}) \mathcal{B}^m(q, \boldsymbol{u}_N(\boldsymbol{\mu})) = \langle G_h(\boldsymbol{\mu}), q \rangle \qquad \text{for all } q \in \mathcal{Q}_h,$$

and the reduced basis equations (20) respectively into

$$(26) \qquad \sum_{m=1}^{M_a} \Theta_a^m(\boldsymbol{\mu}) \mathcal{A}^m(\boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{v}) + \sum_{m=1}^{M_b} \Theta_b^m(\boldsymbol{\mu}) \mathcal{B}^m(p_N(\boldsymbol{\mu}), \boldsymbol{v}) = \langle \boldsymbol{F}_h(\boldsymbol{\mu}), \boldsymbol{v} \rangle \qquad \text{for all } \boldsymbol{v} \in \mathcal{Y}_N$$

$$\sum_{m=1}^{M_b} \Theta_b^m(\boldsymbol{\mu}) \mathcal{B}^m(q, \boldsymbol{u}_N(\boldsymbol{\mu})) = \langle G_h(\boldsymbol{\mu}), q \rangle \qquad \text{for all } q \in \mathcal{Q}_N.$$

Once bases $\{\boldsymbol{\xi}_n^v\}_{n=1}^{2N}$ and $\{\xi_n^p\}_{n=1}^N$ for the reduced basis spaces $\tilde{\mathcal{Y}}_N$ and $\mathcal{Q}_N$ respectively have been constructed, the reduced system matrices

$$(27) \qquad \begin{aligned} [A^m]_{k,\ell} &:= \mathcal{A}^m(\boldsymbol{\xi}_k^v, \boldsymbol{\xi}_\ell^v), \quad 1 \leq k, \ell \leq 2N; \\ [B^m]_{k,\ell} &:= \mathcal{B}^m(\boldsymbol{\xi}_k^v, \xi_\ell^p), \quad 1 \leq k \leq 2N, \quad 1 \leq \ell \leq N, \end{aligned}$$

are independent of the parameters $\boldsymbol{\mu}$, and can be precomputed and stored. This is the parameter-independent *offline stage*. In the *online stage*, for any $\boldsymbol{\mu} \in \mathcal{D}$ we assemble and solve the $3N \times 3N$ system for the reduced velocity $\boldsymbol{u}_N \in \mathbb{R}^{2N \times 1}$ and reduced pressure $\boldsymbol{p}_N \in \mathbb{R}^{N \times 1}$ s.t.

$$(28) \qquad \left(\sum_{m=1}^{M_a} \Theta_a^m(\boldsymbol{\mu}) A^m \right) \boldsymbol{u}_N + \left(\sum_{m=1}^{M_b} \Theta_b^m(\boldsymbol{\mu}) B^m \right) \boldsymbol{p}_N = \boldsymbol{F}(\boldsymbol{\mu})$$

$$\sum_{m=1}^{M_b} \Theta_b^m(\boldsymbol{\mu}) [B^m]^T \boldsymbol{u}_N = \boldsymbol{G}(\boldsymbol{\mu}).$$

In the case of pure Dirichlet boundary conditions we have

$$(29) \qquad \boldsymbol{F}(\boldsymbol{\mu}) = \boldsymbol{f}^F - \sum_{m=1}^{M_a} \Theta_a^m(\boldsymbol{\mu}) A^m \tilde{\boldsymbol{u}}_0$$
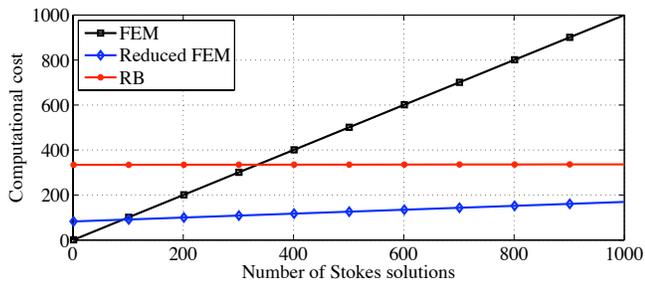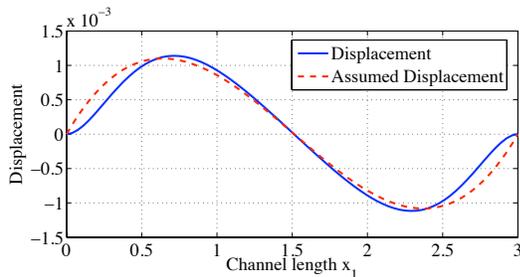
(a) Displacement η and assumed displacement $\hat{\eta}$ at end of coupling Algorithm 1 for (12)

(b) Computational costs of FEM, reduced FEM, and reduced basis approach as number of parametric PDE solutions increases

FIGURE 3. Result of the fluid-structure algorithm and a comparison of computational costs

where the inhomogeneous Dirichlet condition has been treated by lifting the solution of the homogeneous equation into the space with proper boundary conditions. Therefore also the computation of the right-hand side splits affinely in terms of parametric dependence. The assembly and solution of the reduced system can be done independent (and thus very efficiently) of the dimensions of the finite element spaces for the velocity $\mathcal{Y}_h$ and pressure $\mathcal{Q}_h$ respectively. While the offline stage is more expensive compared to solving the finite element problem, if the parametric PDE is evaluated for sufficiently many $\boldsymbol{\mu}$ (typically the cutoff point is between 100-500 PDE evaluations), the inexpensive online stage negates the preliminary costs involved in setting up the reduced basis matrices.

In our case the viscous transformation tensor ν_T obtained by free-form deformations does not satisfy the affine parameterization assumption (24). This difficulty is handled by using the Empirical Interpolation Method (EIM) [1, 9, 19], which approximates the nonaffinely parameterized tensor with a suitable affinely parameterized one by replacing each term of the viscous transformation tensor with an approximate expansion

$$(30) \qquad [\nu_T]_{i,j} = \sum_{m=1}^{M_{i,j}} \vartheta_m^{i,j}(\boldsymbol{\mu})\zeta_m^{i,j}(\boldsymbol{x}) + \varepsilon^{i,j}(\boldsymbol{x}, \boldsymbol{\mu}),$$

where the error terms are chosen to be under some tolerance, $||\varepsilon^{i,j}(\cdot, \boldsymbol{\mu})||_{L^\infty} < \varepsilon_{tol}^{EIM}$ for all $\boldsymbol{\mu} \in \mathcal{D}$. For this problem we use $\varepsilon_{tol}^{EIM} = 1\text{e-}5$, which is satisfied with $M_{i,j} \leq 22$ for $i, j = 1, 2$. The affine decomposition (24) is then recovered. The tolerance should be chosen small enough so that the introduced additional error term does not dominate the error reduced basis approximation [12, 19]. For an a posteriori error estimate of reduced basis approximation for the Stokes problem we refer at the moment to [18].

6. NUMERICAL EXPERIMENTS

Because an analytical solution for the fluid-structure problem described in Sect. 2 is not obtainable, we verified our (finite element) computations by comparing them to results presented in [10] for the same test case and parameter values, except with a second-order wall law. For a flexible channel with length 3 cm and half-width 0.5 cm we imposed a Poiseuille flow profile at both the inflow and outflow

$$(31) \qquad \boldsymbol{u}_0(\boldsymbol{x}) = v_0 \begin{bmatrix} 1 - 4x_2^2 \\ 0 \end{bmatrix}.$$

The numerical values for the physical parameters were chosen according to [3] so that the inflow velocity was $v_0 = 30$ cm/s and the blood viscosity $\nu = 0.035$ g/cm·s. The volume force on the fluid $\boldsymbol{f}^F$ was taken to be zero. The spring constant was $K = 62.5$ g/s^2. From the initial guess $\boldsymbol{\mu} = \boldsymbol{0}$ the least-squares optimization algorithm for the strong parametric coupling took 287 iteration steps (Stokes evaluation + solution of least-squares problem for the coupling) until the stopping criteria $||\boldsymbol{\mu}^{k+1} - \boldsymbol{\mu}^k|| < 1\text{e-}5$ was fulfilled. The final value of the cost functional was $J = 5\text{e-}8$. The obtained displacement η of the fluid-structure interface versus the assumed displacement $\hat{\eta}$ are shown in Fig. 3(a).

Once the full FEM solution was verified to be correct, we performed the model reduction of the Stokes problem. We consider two levels of model reduction. In the first level, the problem is parameterized

with free-form deformations following Sect. 3 , the transformation tensors are affinely decomposed with the empirical interpolation as in (30), and the parametric decomposition (25) of the system matrices is precomputed. Then we can either:

(1) Assemble and solve the full FEM Stokes-problem from (25) using the precomputed system matrices for each $\boldsymbol{\mu}$. The only reduction is in terms of the parameterized geometry. We call this *reduced FEM*.

(2) Perform the full reduced basis offline stage and construct the reduced system matrices in (26), then solve the reduced basis system for each $\boldsymbol{\mu}$. Reduction is both in terms of the parameterized geometry and the PDEs. This is the *reduced basis* approach.

The computational costs are compared in Fig. 3(b). The costs are scaled so that the cost of one full FEM solution equals 1. The full FEM incurs no starting overhead, but scales poorly as the number of PDE solutions increases. The reduced FEM has moderate starting cost (empirical interpolation + matrix assembly) and scales fairly well. The full reduced basis approach is costly in terms of the offline stage, but solving the PDE is almost free, resulting in extremely good scalability when thousands of parametric PDE evaluations are desired. For the steady FSI problem we don't need thousands of PDE evaluations, so that reduced FEM would seem the best option in terms of total computational cost for this problem. For the unsteady case we anticipate a need for many more PDE solutions during time-stepping so that the added scalability of the full reduced basis becomes significant. This is future work to be considered.

7. CONCLUSIONS

A model reduction technique for steady fluid-structure interaction problem of Stokes flow in a flexible 2-d channel was proposed. A geometric reduction was performed using free-form deformations to reduce the free-boundary problem to a low-dimensional parameter space. Reduced basis methods were then used to reduced the complexity of the resulting parametric PDEs.

A least-squares parametric coupling formulation between the fluid and structure was given. The approximate balance of stresses across the interface was formulated using the parameterized displacement of the interface and solved using nonlinear programming techniques. A low-dimensional parameterization with six parameters coming from a free-form deformation technique was enough to obtain approximate coupling.

Computational costs between the full FEM, a geometrically reduced FEM, and the full reduced basis approach were compared. It was observed that the reduced FEM without the overhead of the reduced basis reduction was the computationally most attractive choice, but that for unsteady or nonlinear problems there could be a need for the better scalability of reduced basis methods as the number of PDE evaluations increases.

ACKNOWLEDGEMENTS

REFERENCES

[1] M. Barrault, Y. Maday, N.C. Nguyen, and A.T. Patera. An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations. *C. R. Math. Acad. Sci. Paris*, 339(9):667 – 672, 2004.

[2] S. Deparis. Reduced basis error bound computation of parameter-dependent Navier– Stokes equations by the natural norm approach. *SIAM J. Num. Anal.*, 46(4):2039–2067, 2008.

[3] L. Formaggia, J.-F. Gerbeau, F. Nobile, and F. Quarteroni. On the coupling of 3D and 1D Navier-Stokes equations for flow problems in compliant vessels. *Comput. Methods Appl. Mech. Engrg.*, 191(6-7):561–582, 2001.

[4] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations: Theory and algorithms*. Springer-Verlag Berlin and New York, 1986.

[5] C. Grandmont. Existence et unicité de solutions d'un problème de couplage fluide-structure bidimensionnel stationnaire. *C. R. Math. Acad. Sci. Paris*, 326:651–656, 1998.

[6] M.A. Grepl and A.T. Patera. A posteriori error bounds for reduced-basis approximations of parametrized parabolic partial differential equations. *ESAIM Math. Modelling Numer. Anal.*, 39(1):157–181, 2005.

[7] D.B.P. Huynh, N.C. Nguyen, A.T. Patera, and G. Rozza. Rapid reliable solution of the parametrized partial differential equations of continuum mechanics and transport, 2008. http://augustine.mit.edu.

[8] A.E. Løvgren, Y. Maday, and E.M. Rønquist. A reduced basis element method for the steady Stokes problem. *ESAIM Math. Modelling Numer. Anal.*, 40(3):529–552, 2006.

[9] Y. Maday, N.C. Nguyen, A.T. Patera, and G.S.H. Pau. A general multipurpose interpolation procedure: the magic points. *Commun. Pure Appl. Anal.*, 8(1), 2009.

[10] C.M. Murea. The BFGS algorithm for a nonlinear least squares problem arising from blood flow in arteries. *Comput. Math. Appl.*, 49:171–186, 2005.

[11] C.M. Murea. Numerical simulation of a pulsatile flow through a flexible channel. *ESAIM Math. Modelling Numer. Anal.*, 40(6):1101–1125, 2006.

[12] N.C. Nguyen. A posteriori error estimation and basis adaptivity for reduced-basis approximation of nonaffine-parametrized linear elliptic partial differential equations. *J. Comp. Phys.*, 227:983–1006, 2007.

[13] F. Nobile. *Numerical approximation of fluid-structure interaction problems with application to haemodynamics*. PhD thesis, École Polytechnique Fédérale de Lausanne, 2001.

[14] A.T. Patera and G. Rozza. *Reduced Basis Approximation and A Posteriori Error Estimation for Parametrized Partial Differential Equation*. Version 1.0, Copyright MIT 2006, to appear in (tentative rubric) MIT Pappalardo Graduate Monographs in Mechanical Engineering, 2009.

[15] A. Quarteroni and L. Formaggia. Mathematical modelling and numerical simulation of the cardiovascular system. *In: Ayache, N. (Ed.), Computational Models for the Human Body, Handbook of Numerical Analysis (P.G Ciarlet Ed.)*, pages 3–129, 2004.

[16] A. Quarteroni and G. Rozza. Numerical solution of parametrized Navier-Stokes equations by reduced basis methods. *Numer. Methods Partial Differential Equations*, 23(4):923–948, 2007.

[17] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations (1st Ed.)*. Springer-Verlag, Berlin-Heidelberg, 1994.

[18] D.V. Rovas. *Reduced-basis output bound methods for parametrized partial differential equations*. PhD thesis, Massachusetts Institute of Technology, 2003.

[19] G. Rozza. Reduced basis methods for Stokes equations in domains with non-affine parameter dependence. *Comput. Vis. Sci.*, 12(1):23–35, 2009.

[20] G. Rozza, D.B.P. Huynh, and A.T. Patera. Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations. *Arch. Comput. Methods Engrg.*, 15:229–275, 2008.

[21] G. Rozza and K. Veroy. On the stability of Reduced Basis methods for Stokes Equations in parametrized domains. *Comput. Methods Appl. Mech. Engrg.*, 196(7):1244–1260, 2007.

[22] T.W. Sederberg and S.R. Parry. Free-form deformation of solid geometric models. *Comput. Graph.*, 20(4), 1986.

[23] K. Veroy and A.T. Patera. Certified real-time solution of the parametrized steady incompressible Navier-Stokes equations: rigorous reduced-basis a posteriori error bounds. *Int. J. Numer. Meth. Fluids*, 47(8-9):773–788, 2005.