Exact soliton-like probability measures for interacting jump processes

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Abstract

The cooperative dynamics of a 1-D collection of Markov jump, interacting stochastic processes is studied via a mean-field (MF) approach. In the time-asymptotic regime, the resulting nonlinear master equation is analytically solved. The nonlinearity compensates jumps induced diffusive behavior giving rise to a soliton-like stationary probability density. The soliton velocity and its sharpness both intimately depend on the interaction strength. Below a critical threshold of the strength of interactions, the cooperative behavior cannot be sustained leading to the destruction of the soliton-like solution. The bifurcation point for this behavioral phase transition is explicitly calculated.

Keywords: jump Markov processes, exact solution of a nonlinear master equation, mean-field description of interacting stochastic processes, soliton-like propagating probability measures.

Classification: 60J75

1 Introduction

Interacting stochastic agents are modeled by a collection of nonlinearly coupled Markovian stochastic processes. Inspired by the dynamics recently exposed in Balázs [1], we focus on pure, right-oriented jump processes. For large and homogeneous swarms, the mean-field description offers a powerful method to characterize the resulting nonlinear global dynamics. Adopting the MF approach, the swarm behavior is summarized into a field density variable obeying a nonlinear master equation.

Such partial differential integral equations are in general barely completely solvable. Nevertheless, several explicitly solvable models have been recently studied Hongler [2], Balázs [1]. Our present goal is to enrich this yet available collection by proposing an intrinsically nonlinear extension of the recent models introduced by Balázs [1]. Models involving pure jumps complete the solvable models with dynamics driven either by Brownian Motion and or by alternating Markov renewal processes Hongler [2]. For strong enough mutual interactions, we explicitly observe the existence of a stationary probability measure propagating like a soliton. This soliton-like dynamics can be formed since the underlying nonlinear mechanism due to interactions exactly compensates the jump induced diffusion. This exhibits a close analogy with nonlinear wave dynamics where nonlinearity compensates the velocity dispersion. Since the model is uni-dimensional, long-range interactions between the agents are mandatory for the existence of cooperative behaviors here described by soliton-like probability measures. Decreasing the strength of the mutual interactions, via a barycentric modulation function similar to the one used in Balázs [1], we reach a critical threshold below which no stable cooperative behavior can be sustained. The critical threshold where the behavioral phase transition occurs can here be exactly calculated.

2 Linear pure jump stochastic processes

Let us first describe the dynamics of a single, isolated jump process which later in section 2, will enter into the composition of our interacting swarm. On $\mathbb{R}$, we consider the right-oriented jump Markovian process $X(t)$ characterized by the (linear) master equation:

$$\partial_t P(x,t) = -P(x,t) + \int_{-\infty}^{x} P(y,t)\varphi(x-y)dy, \tag{1}$$

where $P(x,t)$ with $P(x,0) = f(x)$ stands for the transition probability density. The function $\varphi(x) : \mathbb{R} \to \mathbb{R}^+$ defines the probability density for the (right oriented) lengths of the process jumps.
Taking the x-Laplace transform of Eq.(1) and taking into account of the convolution structure, we obtain directly:

$$\partial_t \tilde{P}(s,t) = -\left[1 - \tilde{\varphi}(s)\right] \tilde{P}(s,t). \tag{2}$$

whose solution reads:

$$\tilde{P}(s,t) = e^{-t+\tilde{\varphi}(s)t}, \tag{3}$$

where in writing Eq.(3), we have already assumed the initial condition:

$$P(x,t)\mid_{t=0}=\delta(x). \tag{4}$$

Example. Consider the dynamics obtained when $\varphi(x)=\lambda e^{-\lambda x}$ yielding $\tilde{\varphi}(s)=\frac{\lambda}{\lambda+s}$ and when the initial probability density is $f(x)=\delta(x)$. Accordingly Eq.(3) reads:

$$\tilde{P}(s,t)=e^{-t}\left[e^{t\left(\frac{\lambda}{\lambda+s}\right)}\right]=e^{-t}\left\{\sum_{n=0}^{\infty}\frac{(\lambda\, t)^n}{n!}\left[\frac{1}{\lambda+s}\right]^n\right\}. \tag{5}$$

The Laplace inversion of Eq.(5) yields:

$$P(x,t)=e^{-t}\left\{\delta(x)+\underbrace{e^{-\lambda x}\sum_{n=1}^{\infty}\frac{(\lambda\, t)^n}{n!}\frac{x^{n-1}}{(n-1)!}}_{:=J(x,t)}\right\}. \tag{6}$$

For $J(x,t)$, we can write:

$$J(x,t)=\frac{d}{dx}\left\{\underbrace{\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\frac{x^n}{n!}}_{\mathbb{I}_0\left(2\sqrt{\lambda\, x\, t}\right)-1}\right\}=\frac{\sqrt{\lambda t}}{\sqrt{x}}\mathbb{I}_1\left(2\sqrt{\lambda\, x\, t}\right) \tag{7}$$

where $\mathbb{I}_m(z)$ stands for the m-modified Bessel's functions.
Hence the final probability density $P(x,t)$ reads:

$$P(x,t)=e^{-t}\left\{\delta(x)+e^{-\lambda x}\frac{\sqrt{\lambda t}}{\sqrt{x}}\mathbb{I}_1\left(2\sqrt{\lambda\, x\, t}\right)\right\},\quad x\in\mathbb{R}^+ \tag{8}$$

and one may explicitly verify that one indeed has; $\int_{\mathbb{R}^+}P(x,t)dx=1$, (use the entry 6.643(2) in Gradshteyn [3]).
For time asymptotic regimes, Eq.(8) behaves as:

$$\lim_{t\to\infty}P(x,t)\simeq\frac{(\lambda\, t)^{\frac{1}{4}}}{2\sqrt{\pi}x^{\frac{3}{4}}}e^{-\left[\sqrt{\lambda x}-\sqrt{t}\right]^2}, \tag{9}$$

exhibiting therefore a diffusive propagating wave with vanishing amplitude and velocity $V:=\frac{1}{\lambda}$. Due to translation invariance of the dynamics, we note that $P(x-y,t)$ fulfills a $\delta(x-y)$ initial condition.

Hence, when $P(x,0) = f(x)$, the linearity of the dynamics Eq.(1) enables us to write:

$$\begin{cases} P_f(x,0) = f(x), \\ \\ P_f(x,t) = \int_{\mathbb{R}+} P((x-y),t) f(y) dy. \end{cases} \tag{10}$$

3 Non-linear Markovian jump processes

Keeping the jumps probability density as $\varphi(x) = \lambda e^{-\lambda x}$, let us now consider a large homogeneous collection of identical processes evolving like Eq.(1) now subject to mutual long-range interactions. The class of interactions we consider yields, in the mean-field limit, the nonlinear master equation:

$$\begin{cases} \Omega(x,t) = \int_x^\infty g\left(z - \langle X(t)\rangle\right) \partial_z G(z,t) dz \\ \\ \partial_{xt} G(x,t) = -\Omega(x,t)\partial_x G(x,t) + \int_{-\infty}^x \Omega(y,t)\partial_y G(y,t) \lambda e^{-\lambda(x-y)} dy, \\ \\ \langle X(t)\rangle = \int_{\mathbb{R}+} y\, \partial_y G(y,t)\, dy, \end{cases} \tag{11}$$

where $G(x,t)$ stands for the cumulative distribution of the a nonlinear jump process, (i.e. $G(x,t)$ is monotonically increasing with boundary conditions $G(-\infty,t) = 0$ and $G(\infty,t) = 1$). Note that while in Eq.(1) the jumping rate is unity, in Eq.(11) it is replaced by $\Omega(x,t) > 0$ which is explicitly state-dependent. This is precisely where the mutual interaction introduces a strong nonlinearity into the dynamics. In the sequel, we focus on cases where $g(x) = g(-x) > 0$.

For asymptotic time, we now postulate that Eq.(11) admits ξ-functional dependent solutions with $\xi = (x - Vt)$ and with the even symmetry:

$$\int_{\mathbb{R}} \xi \partial_\xi G(\xi) d\xi = 0, \tag{12}$$

where V is a propagating velocity parameter. In terms of ξ, Eq.(11) can be rewritten as:

$$V\left[\partial^3_{\xi\xi\xi} G(\xi) + \lambda \partial^2_{\xi\xi} G(\xi)\right] = \partial_\xi \left\{\Omega(\xi)\partial_\xi G(\xi)\right\}. \tag{13}$$

Defining $\mathcal{L}(\xi) := \log\left[\partial_\xi G(\xi)\right]$, after one integration step where the integration constant is taken to be zero, Eq.(13) can be rewritten as:

$$V\partial^2_{\xi\xi}\mathcal{L}(\xi) = -\lambda V + \int_\xi^\infty \left[g(\eta)\partial_\eta G(\eta)\right] d\eta. \tag{14}$$

Assuming now a functional dependence $g(\xi) = \cosh^{-n}(\xi)$ with $n \in \mathbb{R}$, by direct substitution, one can immediately see that Eq.(14) is solved by the (normalized) probability density $\partial_\xi G(\xi)$:

$$\begin{cases} \partial_\xi G(\xi) = \dfrac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)} \cosh^{-m}(\xi), \qquad m > 0, \\ \\ m = \lambda = 2 - n, \\ \\ V = \dfrac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)}. \end{cases} \tag{15}$$

Due to the ξ-symmetry of the probability density $\partial_\xi G(\xi)$, Eq.(12) is trivially satisfied.
For $n \in]2, -\infty]$, Eq.(15) implies that a stationary propagating density $\partial_x G(x)$ is sustained by the nonlinear dynamics Eq.(11). However, for short decaying $g(x)$-modulation, occurring when $n > 2$, no stationary propagating probability density exists, (i.e. for this parameter range, $m < 0$ in Eq.(15) and the solution cannot be normalized to unity as required for a probability measure). For this exactly solvable dynamics, we also observe that the average jump length λ^{-1} and the barycentric modulation strength controlled by the factor n are intimately dependent control parameters. In addition, we note that for large m, the asymptotic expansion of the Γ-function implies that $\lim_{m\to\infty} V \simeq \sqrt{m}$.

Illustration. Along the same lines as in Hongler [2], the nonlinear dynamics given by Eq.(11) can be viewed as representing the mean-field evolution associated with a large population of stochastic jumping agents subject to a mutual imitation process. The swarm dynamics is described via the probability density function $\partial_x G(x,t)$ obeying a nonlinear partial differential equations (PDE). Mutual interactions of agents are responsible for the state-dependent jumping rate $\Omega(x,t)$ in Eq.(11). The functional form of $\Omega(x,t)$ simultaneously includes two distinct nonlinear features, namely:

a) **imitation process**. To isolate this process, we may consider the case $g(x) \equiv 1$, (i.e. $n = 0$) implying that

$$\Omega(x,t) = 1 - G(x,t). \tag{16}$$

The resulting state-dependent jumping rate Eq.(16) induces a traveling and compacting tendency. As the agents are subject to pure right-oriented jumps, Eq.(16) effectively describes situations where the laggard agents jump more frequently than the leaders, (i.e. laggards try to effectively imitate the leaders' behavior).

b) **barycentric range modulation of the mutual interactions**. The modulation obtained when $g(x) \neq 1$ describes the relative importance attributed to interactions with agents remote from the barycenter $\langle X(t) \rangle$ of the swarm. Here, we may separate two distinct tendencies:

i) when $n \in [0, 2[$, far remote agents tend not to influence the dynamics. In this case, the resulting behavior can be referred as a **weak cooperative identity** and the propagating probability density given by Eq.(15) exhibits the shape of **a table-top soliton** with a plateau increasing when the limiting value 2 is approached. One observes a comparatively low propagating velocity V of these table-top like aggregates. Again, we emphasize that for $n > 2$, the cooperative interactions are not strong enough to sustain the propagation of a cooperative behavior in asymptotic time. This is well known in general for 1-D stochastic interacting system, (the Ising model being the paradigmatic example) where no cooperative phase can be formed when the interactions operate on too limited ranges.

ii) for $n < 0$, the $g(x)$ modulation effectively gives rise to a **strong cooperative identity**. Far remote agents increasingly influence the swarm. This gives rise to sharply peaked solitons-like probability densities propagating with high velocities.

References

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