A CANONICAL ENRICHED ADAMS-HILTON MODEL FOR SIMPLICIAL SETS

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Abstract. For any 1-reduced simplicial set K we define a canonical, coassociative coproduct on $\Omega C(K)$, the cobar construction applied to the normalized, integral chains on K, such that any canonical quasi-isomorphism of chain algebras from $\Omega C(K)$ to the normalized, integral chains on GK, the loop group of K, is a coalgebra map up to strong homotopy. Our proof relies on the operadic description of the category of chain coalgebras and of strongly homotopy coalgebra maps given in [HPS].

Introduction

Let X be a topological space. It is, in general, quite difficult to calculate the algebra structure of the loop space homology $H_*\Omega X$ directly from the (singular or cubical) chain complex $C_*\Omega X$. An algorithm that associates to a space X a differential graded algebra whose homology is relatively easy to calculate and isomorphic as an algebra to $H_*\Omega X$ is therefore of great value.

In 1955 [AH], Adams and Hilton invented such an algorithm for the class of simply-connected CW-complexes, which can be summarized as follows. Let X be a CW-complex such that X has exactly one 0-cell and no 1-cells, and such that every attaching map is based with respect to the unique 0-cell of X. There exists a morphism of differential graded algebras inducing an isomorphism on homology—a *quasi-isomorphism*—

$$\theta_X : (TV, d) \xrightarrow{\simeq} C_*\Omega X,$$

such that θ_X restricts to quasi-isomorphisms $(TV_{\leq n}, d) \xrightarrow{\simeq} C_*\Omega X_{n+1}$, where X_{n+1} denotes the $(n+1)$-skeleton of X, TV denotes the free (tensor) algebra on a free,

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graded $\mathbb{Z}$-module V, ΩX is the space of Moore loops on X and C_* denotes the cubical chains. The morphism θ_X is called an *Adams-Hilton model of* X and satisfies the following properties.

- If $X = *\cup\bigcup_{\alpha\in A} e^{n_\alpha+1}$, then V has a degree-homogeneous basis $\{v_\alpha : \alpha \in A\}$ such that $\deg v_\alpha = n_\alpha$.
- If $f_\alpha : S^{n_\alpha} \to X_{n_\alpha}$ is the attaching map of the cell $e^{n_\alpha+1}$, then $[\theta(dv_\alpha)] = \mathcal{K}_{n_\alpha}[f_\alpha]$. Here, $\mathcal{K}_{n_\alpha}$ is defined so that

$$\begin{array}{ccc} \pi_{n_\alpha}X_{n_\alpha} & \xrightarrow{\cong} & \pi_{n_\alpha-1}\Omega X_{n_\alpha} \\ & \searrow^{\mathcal{K}_{n_\alpha}} & \downarrow h \\ & & H_{n_\alpha-1}\Omega X_{n_\alpha} \end{array}$$

  commutes, where h denotes the Hurewicz homomorphism.

It follows that (TV, d) is unique up to isomorphism.

The Adams-Hilton model has proved to be a powerful tool for calculating the loop space homology algebra of CW-complexes. Many common spaces have Adams-Hilton models that are relatively simple and thus well-adapted to computations. Difficulties start to arise, however, when one wishes to use the Adams-Hilton model to compute the algebra homomorphism induced by a cellular map $f : X \to Y$.

If $\theta_X : (TV, d) \to C_*\Omega X$ and $\theta_Y : (TW, d) \to C_*\Omega Y$ are Adams-Hilton models, then there exists a unique homotopy class of morphisms $\varphi : (TV, d) \to (TW, d)$ such that

$$\begin{array}{ccc} (TV,d) & \xrightarrow{\varphi} & (TW,d) \\ \downarrow \theta_X & & \downarrow \theta_Y \\ C_*\Omega X & \xrightarrow{C_*\Omega f} & C_*\Omega Y \end{array}$$

commutes up to derivation homotopy. Any representative φ of this homotopy class can be said to be an Adams-Hilton model of f.

As the choice of φ is unique only up to homotopy, the Adams-Hilton model is not a functor. The essential problem is that choices are made at each stage of the construction of θ_X and θ_Y: they are not canonical. For many purposes this lack of functoriality does not cause any problems. When one needs to use Adams-Hilton models to construct new models, however, then it can become quite troublesome, as seen in, e.g., [DH] .

Similarly, when constructing algebraic models based on Adams-Hilton models, one often needs the models to be *enriched*, i.e., there should be a chain algebra map $\psi : (TV, d) \to (TV, d) \otimes (TV, d)$ such that

$$\begin{array}{ccc} (TV,d) & \xrightarrow{\psi} & (TV,d)\otimes(TV,d) \\ \downarrow \theta_X & & \downarrow \theta_X\otimes\theta_X \\ C_*\Omega X & \xrightarrow{AW\circ C_*\Omega\Delta_X} & C_*\Omega X\otimes C_*\Omega X \end{array}$$

commutes up to homotopy, where AW denotes the Alexander-Whitney equivalence. Thus θ_X is a coalgebra map up to homotopy. Since the underlying algebra of (TV, d)

is free, such a coproduct always exists and can be constructed degree by degree. Again, however, choices are involved in the construction of ψ, so that one usually knows little about it, other than that it exists. In particular, since the diagram above commutes and $AW\circ C_*\Omega\Delta_X$ is cocommutative up to homotopy and strictly coassociative, ψ is coassociative and cocommutative up to homotopy, i.e., (TV,d,ψ) is a *Hopf algebra up to homotopy* [A]. For many constructions, however, it would be very helpful to know that there is a choice of ψ that is strictly coassociative.

Motivated by the need to rigidify the Adams-Hilton model construction and its enrichment, we work here with simplicial sets rather than topological spaces. Any topological space X that is equivalent to a finite-type simplicial complex is homotopy-equivalent to the realization of a finite-type simplicial set. There is an obvious candidate for a canonical Adams-Hilton model of a 1-reduced simplicial set K: $\Omega C(K)$, the cobar construction on the integral, normalized chains on K, which is a free algebra on generators in one-to-one correspondence with the nondegenerate simplices of K. It follows easily by acyclic models methods (see, e.g, [M]) that there exists a natural quasi-isomorphism of chain algebras

$$\theta_K:\Omega C(K)\xrightarrow{\simeq}C(GK),$$

where GK denotes the Kan loop group on K. There is also an explicit formula for such a natural transformation, due to Szczarba [Sz].

In this article we provide a simple definition of a natural, strictly coassociative coproduct, the *Alexander-Whitney (A-W) cobar diagonal*,

$$\psi_K:\Omega C(K)\to\Omega C(K)\otimes\Omega C(K),$$

where K is any 1-reduced simplicial set. Furthermore, any natural quasi-isomorphism of chain algebras $\theta_K:\Omega C(K)\xrightarrow{\simeq}C(GK)$ is a strongly homotopy coalgebra map with respect to ψ_K. In other words, θ_K is a coalgebra map up to homotopy; the homotopy in question is a coderivation up to a second homotopy; etc. The map θ_K has already proved extremely useful in constructing a number of interesting algebraic models, such as in [BH],[H], [HL].

Ours is not the only definition of a canonical, coassociative coproduct on $\Omega C(K)$. In [Ba] Baues defined combinatorially an explicit coassociative coproduct $\widetilde{\psi}_K$ on $\Omega C(K)$, together with an explicit derivation homotopy insuring cocommutativity up to homotopy. He showed that there is an injective quasi-isomorphism of chain Hopf algebras from $(\Omega C(K),\widetilde{\psi}_K)$ into (the first Eilenberg subcomplex of) the cubical cochains on the geometrical cobar construction on K.

We show in section 5 of this article that Baues's coproduct is equal to the Alexander-Whitney cobar diagonal, a result that is surprising at first sight. It is clear from the definition of Baues's coproduct that its image lies in $\Omega C(K)\otimes s^{-1}C_+(K)$, so that its form is highly asymmetric. That asymmetry is well hidden in our definition of the Alexander-Whitney cobar diagonal.

Even though the two definitions are equivalent, our approach is still interesting, as the Alexander-Whitney cobar diagonal is given explicitly in terms of only two fundamental pieces: the diagonal map on a simplicial set and the Eilenberg-Zilber strong deformation retract

$$C(K)\otimes C(K)\underset{f}{\overset{\nabla}{\rightleftarrows}}C(K\times K)\circlearrowleft\varphi.$$

(See section 2.) Furthermore, it is very helpful for construction purposes to have an explicit equivalence $\theta_K : \Omega C(K) \longrightarrow C(GK)$ that is a map of coalgebras up to strong homotopy and a map of algebras, as the articles [BH],[H], [HL] amply illustrate.

In a subsequent article [HPS2], we will further demonstrate the importance of our coproduct definition on the cobar construction, when we treat the special case of suspensions. In particular we will show that the Szczarba equivalence is a strict coalgebra map when K is a suspension.

After recalling a number of elementary definitions at the end of this introduction, we devote section 1 to Gugenheim and Munkholm's category **DCSH**, the category of chain coalgebras and strongly homotopy coalgebra maps. In particular, we recall and expand upon the "operadic" description of **DCSH** developed in [HPS]. In section 2 we introduce homological perturbation theory and its interaction with morphisms in **DCSH**, expanding the discussion to include twisting cochains and twisting functions in section 3. The heart of this article is section 4, where we define the Alexander-Whitney cobar diagonal, show that it is cocommutative up to homotopy and strictly coassociative, and prove that the Szczarba equivalence is a strongly homotopy coalgebra map. We conclude section 4 with a discussion of the relationship of our work to the problem of iterating the cobar construction. In section 5 we prove that the Alexander-Whitney cobar diagonal is equal to Baues's coproduct on $\Omega C(K)$.

In a forthcoming paper we will explain how the canonical Adams-Hilton model enables us to carry out Bockstein spectral sequence calculations using methods previously applied only in the "Anick" range (cf., [Sc]) to spaces well outside of that range.

Preliminary definitions, terminology and notation.

We recall here certain necessary elementary definitions and constructions. We also introduce notation and terminology that we use throughout the remainder of this paper.

We consider that the set of natural numbers $\mathbb{N}$ includes 0.

If $\mathbf{C}$ is a category and A and B are objects in $\mathbf{C}$, then $\mathbf{C}(A, B)$ denotes the collection of morphisms from A to B. We write $\mathbf{C}^{\rightarrow}$ for the category of morphisms in $\mathbf{C}$.

Given chain complexes (V, d) and (W, d), the notation $f : (V, d) \xrightarrow{\simeq} (W, d)$ indicates that f induces an isomorphism in homology. In this case we refer to f as a *quasi-isomorphism*.

The *suspension* endofunctor s on the category of graded modules is defined on objects $V = \bigoplus_{i\in\mathbb{Z}} V_i$ by $(sV)_i \cong V_{i-1}$. Given a homogeneous element v in V, we write sv for the corresponding element of sV. The suspension s admits an obvious inverse, which we denote s^{-1}.

A graded R-module $V = \bigoplus_{i\in\mathbb{Z}} V_i$ is *connected* if $V_{<0} = 0$ and $V_0 \cong R$. It is *simply connected* if, in addition, $V_1 = 0$. We write V_+ for $V_{>0}$.

Let V be a positively-graded R-module. The free associative algebra on V is denoted TV, i.e.,

$$TV \cong R \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

A typical basis element of TV is denoted $v_1 \cdots v_n$, i.e., we drop the tensors from the notation. We say that $v_1 \cdots v_n$ is *of length* n and let $T^n V = V^{\otimes n}$ be the submodule of words of length n. The product on TV is then defined by

$$\mu(u_1\cdots u_m\otimes v_1\cdots v_n)=u_1\cdots u_m v_1\cdots v_n.$$

Throughout this paper $\pi : T^{>0}V \longrightarrow V$ denotes the projection map such that $\pi(v_1\cdots v_n)=0$ if $n>1$ and $\pi(v)=v$ for all $v\in V$. When we refer to the *linear part* of an algebra map $f:TV\longrightarrow TW$, we mean the composite $\pi\circ f|_V:V\longrightarrow W$.

Definition. Let (C,d) be a simply-connected chain coalgebra with reduced coproduct $\overline{\Delta}$. The *cobar construction* on (C,d), denoted $\Omega(C,d)$, is the chain algebra $(Ts^{-1}(C_+),d_\Omega)$, where $d_\Omega=-s^{-1}ds+(s^{-1}\otimes s^{-1})\overline{\Delta}s$ on generators.

Observe that for every pair of simply-connected chain coalgebras (C,d) and (C',d') there is a natural quasi-isomorphism of chain algebras

$$q:\Omega\big((C,d)\otimes(C',d')\big)\longrightarrow\Omega(C,d)\otimes\Omega(C',d') \tag{0.1}$$

specified by $q\big(s^{-1}(x\otimes 1)\big)=s^{-1}x\otimes 1$, $q\big(s^{-1}(1\otimes y)\big)=1\otimes s^{-1}y$ and $q\big(s^{-1}(x\otimes y)\big)=0$ for all $x\in C_+$ and $y\in C'_+$ [Mi:Thm. 7.4].

Definition. Let $f,g:(A,d)\longrightarrow(B,d)$ be two maps of chain algebras. An *(f,g)-derivation* is a linear map $\varphi:A\longrightarrow B$ of degree $+1$ such that $\varphi\mu=\mu(\varphi\otimes g+f\otimes\varphi)$, where μ denotes the multiplication on A and B. A *derivation homotopy* from f to g is an (f,g)-derivation φthat satisfies $d\varphi+\varphi d=f-g$.

If f and g are maps of chain coalgebras, there is an obvious dual definition of an *(f,g)-coderivation* and of *(f,g)-coderivation homotopy*.

Definition. Let K be a simplicial set, and let $\mathcal{F}_{ab}$ denote the free abelian group functor. For all $n>0$, let $DK_n=\cup_{i=0}^{n-1}s_i(K_{n-1})$, the set of degenerate n-simplices of K. The *normalized chain complex* on K, denoted $C(K)$, is given by

$$C_n(K)=\mathcal{F}_{ab}(K_n)/\mathcal{F}_{ab}(DK_n).$$

Given a map of simplicial sets $f:K\longrightarrow L$, the induced map of normalized chain complexes is denoted $f_\sharp$.

Definition. Let K be a reduced simplicial set, and let $\mathcal{F}$ denote the free group functor. The *loop group* GK on K is the simplicial group such that $(GK)_n=\mathcal{F}(K_{n+1}\smallsetminus\operatorname{Im}s_0)$, with faces and degeneracies specified by

$$\begin{aligned}\partial_0\bar{x}&=(\,\overline{\partial_0 x}\,)^{-1}\overline{\partial_1 x}\\ \partial_i\bar{x}&=\overline{\partial_{i+1}x}\quad\text{for all } i>0\\ s_i\bar{x}&=\overline{s_{i+1}x}\quad\text{for all } i\geq 0\end{aligned}$$

where $\bar{x}$ denotes the class in $(GK)_n$ of $x\in K_{n+1}$.

Observe that for each pair of reduced simplicial sets (K,L) there is a unique homomorphism of simplicial groups $\rho:G(K\times L)\longrightarrow GK\times GL$, which is specified by $\rho\big(\,\overline{(x,y)}\,\big)=(\bar{x},\bar{y})$.

1. The category DCSH and its relatives

The categor **DCSH** of coassociative chain coalgebras and of coalgebra morphisms up to strong homotopy was first defined by Gugenheim and Munkholm in the early 1970's [GM], when they were studying extended naturality of the functor Cotor. The objects of **DCSH** have a relatively simple algebraic description, while that of the morphisms is rich and complex. Its objects are augmented, coassociative chain coalgebras, and a morphism from C to C' is a map of chain algebras $\Omega C \to \Omega C'$.

In a slight abuse of terminology, we say that a chain map between chain coalgebras $f : C \to C'$ is a *DCSH map* if there is a morphism in $\mathbf{DCSH}(C, C')$ of which f is the linear part. In other words, there is a map of chain algebras $g : \Omega C \to \Omega C'$ such that

$$g|_{s^{-1}C_+} = s^{-1} f s + \text{higher-order terms}.$$

In a further abuse of notation, we sometimes write $\widetilde{\Omega} f : \Omega C \to \Omega C'$ to indicate one choice of chain algebra map of which f is the linear part.

It is also possible to broaden the definition of coderivation homotopy to homotopy of DCSH maps. Given two DCSH maps $f, f' : C \to C'$, a *DCSH homotopy* from f to f' is a $(\widetilde{\Omega} f, \widetilde{\Omega} f')$-derivation homotopy $h : \Omega(C, d) \to \Omega(C', d')$. We sometimes abuse terminology and refer to the linear part of h as a DCSH homotopy from f to f'.

The category **DCSH** plays an important role in topology. For any reduced simplicial set K, the usual coproduct on $C(K)$ is a DCSH map, as we explain in detail in section 2. Furthermore, we show in section 4 that given any natural, strictly coassociative coproduct on $\Omega C(K)$, any natural map of chain algebras $\Omega C(K) \to C(GK)$ is also a DCSH map.

In [HPS] the authors provided a purely operadic description of **DCSH**. Before recalling and elaborating upon this description, we briefly explain the framework in which it is constructed. We refer the reader to section 2 of [HPS] for further details.

Let $\mathbf{M}$ denote the category of chain complexes over a PID R, and let $\mathbf{M}^\Sigma$denote the category of symmetric sequences of chain complexes. An object $\mathcal{X}$ of $\mathbf{M}^\Sigma$ is a family $\{\mathcal{X}(n) \in \mathbf{M} \mid n \geq 0\}$ of objects in $\mathbf{M}$ such that $\mathcal{X}(n)$ admits a right action of the symmetric group Σ_n, for all n. There is a faithful functor $\mathcal{T} : \mathbf{M} \longrightarrow \mathbf{M}^\Sigma$ where, for all n, $\mathcal{T}(A)(n) = A^{\otimes n}$, where Σ_n acts by permuting the tensor factors. The functor $\mathcal{T}$ is strong monoidal, with respect to the *level monoidal structure* $(\mathbf{M}^\Sigma, \otimes, \mathcal{C})$, where $(\mathcal{X} \otimes \mathcal{Y})(n) = \mathcal{X}(n) \otimes \mathcal{Y}(n)$, endowed with the diagonal action of Σ_n, and $\mathcal{C}(n) = R$, endowed with the trivial Σ_n-action.

The category $\mathbf{M}^\Sigma$ also admits a nonsymmetric, right-closed monoidal structure $(\mathbf{M}^\Sigma, \diamond, \mathcal{J})$, where $\diamond$ is the *composition product* of symmetric sequences, and $\mathcal{J}(1) = R$ and $\mathcal{J}(n) = 0$ otherwise. Given symmetric sequences $\mathcal{X}$ and $\mathcal{Y}$, $(\mathcal{X} \diamond \mathcal{Y})(0) = \mathcal{X}(0) \otimes \mathcal{Y}(0)$ and for $n > 0$,

$$(\mathcal{X} \diamond \mathcal{Y})(n) = \coprod_{\substack{k \geq 1 \\ \vec{\imath} \in I_{k,n}}} \mathcal{X}(k) \underset{\Sigma_k}{\otimes} \big(Y(i_1) \otimes \cdots \otimes Y(i_k)\big) \underset{\Sigma_{\vec{\imath}}}{\otimes} R[\Sigma_n],$$

where $I_{k,n} = \{\vec{\imath} = (i_1, ..., i_k) \in \mathbb{N}^k \mid \sum_j i_j = n\}$ and $\Sigma_{\vec{\imath}} = \Sigma_{i_1} \times \cdots \times \Sigma_{i_k}$, seen as a subgroup of Σ_n. For any objects $\mathcal{X}, \mathcal{X}', \mathcal{Y}, \mathcal{Y}'$ in $\mathbf{M}^\Sigma$, there is an obvious, natural intertwining map

$$\iota : (\mathcal{X} \otimes \mathcal{X}') \diamond (\mathcal{Y} \otimes \mathcal{Y}') \longrightarrow (\mathcal{X} \diamond \mathcal{Y}) \otimes (\mathcal{X}' \diamond \mathcal{Y}') \ . \tag{2.1}$$

An *operad* in $\mathbf{M}$ is a monoid with respect to the composition product. The *associative operad* $\mathcal{A}$ is given by $\mathcal{A}(n) = R[\Sigma_n]$ for all n, endowed with the obvious monoidal structure, induced by permutation of blocks.

Let $\mathcal{P}$ denote any operad in $\mathbf{M}$. A *$\mathcal{P}$-coalgebra* consists of an object C in $\mathbf{M}$. together with an appropriately equivariant and associative family

$$\{C \otimes \mathcal{P}(n) \longrightarrow C^{\otimes n} \mid n \geq 0\}$$

of morphisms in $\mathbf{M}$. The functor $\mathcal{T}$ restricts to a faithful functor

$$\mathcal{T} : \mathcal{P}\text{-}\mathbf{Coalg} \longrightarrow \mathbf{Mod}_{\mathcal{P}}$$

from the category of $\mathcal{P}$-coalgebras to the category of right $\mathcal{P}$-modules.

In [HPS] the authors constructed a free $\mathcal{A}$-bimodule $\mathcal{F}$, called the *Alexander-Whitney bimodule*. As symmetric sequences of graded modules, $\mathcal{F} = \mathcal{A} \diamond \mathcal{S} \diamond \mathcal{A}$, where $\mathcal{S}(n) = R[\Sigma_n] \cdot z_{n-1}$, the free $R[\Sigma_n]$-module on a generator of degree $n-1$. Moreover, $\mathcal{F}$ admits an increasing, differential filtration, given by $F_n\mathcal{F} = \mathcal{A} \diamond \mathcal{S}_n \diamond \mathcal{A}$, where $\mathcal{S}_n(m) = \mathcal{S}(m)$ if $m \leq n$ and $\mathcal{S}_n(m) = 0$ otherwise. More precisely, if $\partial_{\mathcal{F}}$ is the differential on $\mathcal{F}$, then

$$\partial_{\mathcal{F}} z_n = \sum_{0 \leq i \leq n-1} \delta \otimes (z_i \otimes z_{n-i-1}) + \sum_{0 \leq i \leq n-1} z_{n-1} \otimes (1^{\otimes i} \otimes \delta \otimes 1^{\otimes n-i-1}),$$

where $\delta \in \mathcal{A}(2) = R[\Sigma_2]$ is a generator.

The Alexander-Whitney bimodule is endowed with a coassociative, counital coproduct

$$\psi_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F} \underset{\mathcal{A}}{\diamond} \mathcal{F},$$

where $\underset{\mathcal{A}}{\diamond}$ denotes the composition product over $\mathcal{A}$, defined as the obvious coequalizer. In particular,

$$\psi_{\mathcal{F}}(z_n) = \sum_{\substack{1 \leq k \leq n+1 \\ \vec{\imath} \in I_{k,n+1}}} z_{k-1} \otimes (z_{i_1-1} \otimes \cdots \otimes z_{i_k-1})$$

for all $n \geq 0$, where $I_{k,n} = \{\vec{\imath} = (i_1, ..., i_k) \mid \sum_j i_j = n\}$.

Furthermore, $\mathcal{F}$ is a level comonoid, i.e., there is a coassociative, counital coproduct

$$\Delta_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{F},$$

which is specified by

$$\Delta_{\mathcal{F}}(z_n) = \sum_{\substack{1 \leq k \leq n+1 \\ \vec{\imath} \in I_{k,n+1}}} \big(z_{k-1} \otimes (\delta^{(i_1)} \otimes \cdots \otimes \delta^{(i_k)})\big) \otimes \big(\delta^{(k)} \otimes (z_{i_1-1} \otimes \cdots \otimes z_{i_k-1})\big).$$

Here, $\delta^{(i)} \in \mathcal{A}(i)$ denotes the appropriate iterated composition product of $\delta^{(2)} = \delta$.

Let $(\mathcal{A}, \psi_{\mathcal{F}})$-**Coalg** denote the category of which the objects are $\mathcal{A}$-coalgebras (i.e., coassociative and counital chain coalgebras) and where the morphisms are defined by

$$(\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}(C, C') := \mathbf{Mod}_{\mathcal{A}}\big(\mathcal{T}(C) \underset{\mathcal{A}}{\circ} \mathcal{F}, \mathcal{T}(C')\big).$$

Composition in $(\mathcal{A}, \psi_{\mathcal{F}})$-**Coalg** is defined in terms of $\psi_{\mathcal{F}}$. Given $\theta \in (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}(C, C')$ and $\theta' \in (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}(C', C'')$, their composite $\theta'\theta \in (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}(C, C'')$ is given by composing the following sequence of (strict) morphisms of right $\mathcal{A}$-modules.

$$\mathcal{T}(C) \underset{\mathcal{A}}{\diamond} \mathcal{F} \xrightarrow{1_{\mathcal{T}(C)} \underset{\mathcal{A}}{\diamond} \psi_{\mathcal{F}}} \mathcal{T}(C) \underset{\mathcal{A}}{\diamond} \mathcal{F} \underset{\mathcal{A}}{\diamond} \mathcal{F} \xrightarrow{\theta \underset{\mathcal{A}}{\diamond} 1_{\mathcal{F}}} \mathcal{T}(C') \underset{\mathcal{A}}{\diamond} \mathcal{F} \xrightarrow{\theta'} \mathcal{T}(C'').$$

We call $(\mathcal{A}, \psi_{\mathcal{F}})$-**Coalg** the *$(\mathcal{F}, \psi_{\mathcal{F}})$-governed category of $\mathcal{A}$-coalgebras*.

The important properties of the Alexander-Whitney bimodule given below follow immediately from the Cobar Duality Theorem in [HPS].

Theorem 1.1[HPS]**.** *For any category* $\mathbf{D}$*, there is a full and faithful functor, called the induction functor,*

$$\mathrm{Ind} : \big((\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}\big)^{\mathbf{D}} \to \big(\mathcal{A}\text{-}\mathbf{Alg}\big)^{\mathbf{D}}$$

defined on objects by $\mathrm{Ind}(X) = \Omega X$ *for all functors* $X : \mathbf{D} \to (\mathcal{A}, \psi_{\mathcal{F}})$-**Coalg** *and on morphisms by*

$$\mathrm{Ind}(\tau)|_{s^{-1}X} = \sum_{k\geq 1} (s^{-1})^{\otimes k} \tau(- \otimes z_{k-1}) s : s^{-1}X \to \Omega Y$$

for all natural transformations $\tau : X \to Y$*.*

As an easy consequence of Theorem 1.1, we obtain the following result.

Corollary 1.2 [HPS]**.** *The category* **DCSH***, is isomorphic to the* $(\mathcal{F}, \psi_{\mathcal{F}})$*-governed category of coalgebras,* $(\mathcal{A}, \psi_{\mathcal{F}})$-**Coalg***.*

The isomorphism of the corollary above is given by the identity on objects and Ind on morphisms.

Define a bifunctor $\wedge : (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg} \times (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg} \to (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}$ on objects by $C \wedge C' := C \otimes C'$, the usual tensor product of chain coalgebras. Given $\theta \in (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}(C, D)$ and $\theta' \in (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}(C', D')$, we define $\theta \wedge \theta'$ to be the composite of (strict) right $\mathcal{A}$-module maps

$$\begin{array}{ccccc}
\mathcal{T}(C \wedge C') \underset{\mathcal{A}}{\diamond} \mathcal{F} & \xrightarrow{\cong} & \big(\mathcal{T}(C) \otimes \mathcal{T}(C')\big) \underset{\mathcal{A}}{\diamond} \mathcal{F} & \xrightarrow{1 \underset{\mathcal{A}}{\diamond} \Delta_{\mathcal{F}}} & \big(\mathcal{T}(C) \otimes \mathcal{T}(C')\big) \underset{\mathcal{A}}{\diamond} (\mathcal{F} \otimes \mathcal{F}) \\
 & & & & \big\downarrow \iota \\
 & & & & \big(\mathcal{T}(C) \underset{\mathcal{A}}{\diamond} \mathcal{F}\big) \otimes \big(\mathcal{T}(C') \underset{\mathcal{A}}{\diamond} \mathcal{F}\big) \\
 & \searrow^{\theta \wedge \theta'} & & & \big\downarrow \theta \otimes \theta' \\
 & & & & \mathcal{T}(D) \otimes \mathcal{T}(D') \\
 & & & & \big\downarrow \cong \\
 & & & & \mathcal{T}(D \wedge D')
\end{array}$$

where ι is the intertwining map of (2.1). It is straightforward to show that $\wedge$ endows $(\mathcal{A}, \psi_{\mathcal{F}})$-**Coalg** with the structure of a monoidal category.

Lemma 1.3. *The induction functor* $\mathrm{Ind} : (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg} \to \mathcal{A}\text{-}\mathbf{Alg}$ *is comonoidal.*

Proof. Let $q : \Omega(-\otimes -) \to \Omega(-)\otimes\Omega(-)$ denote Milgram's natural transformation (0.1) of functors from $\mathcal{A}$-**Coalg** into $\mathcal{A}$-**Alg**. It is an easy exercise, based on the explicit formula for $\Delta_{\mathcal{F}}$, to prove that

$$q\,\mathrm{Ind}(\theta\wedge\theta') = \big(\mathrm{Ind}(\theta)\otimes \mathrm{Ind}(\theta')\big)q : \Omega(C\otimes C') \to \Omega D\otimes \Omega D'$$

for all $\theta \in (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}(C, D)$ and $\theta' \in (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}(C', D')$. Milgram's equivalence therefore provides us with the desired natural transformation

$$q : \mathrm{Ind}(-\wedge -) \to \mathrm{Ind}(-)\otimes \mathrm{Ind}(-). \quad \square$$

In section 4 of this article we consider objects in the following category related to $(\mathcal{A}, \psi_{\mathcal{F}})$-**Coalg**.

Definition. The objects of the *weak Alexander-Whitney category* **wF** are pairs (C, Ψ), where C is a object in $\mathcal{A}$-**Coalg** and $\Psi \in (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}(C, C\otimes C)$ such that

$$\Psi(-\otimes z_0) : C \to C\otimes C$$

is exactly the coproduct on C, while

$$\mathbf{F}\big((C,\Psi),(C',\Psi')\big) = \{\theta \in (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}(C, C') \mid \Psi'\theta = (\theta\wedge\theta)\Psi\}.$$

An object of **wF** is called a *weak Alexander-Whitney coalgebra*.

As we establish in the next lemma, the cobar construction provides an important link between the weak Alexander-Whitney category and the following category of algebras endowed with coproducts.

Definition. The objects of the *weak Hopf algebra category* **wH** are pairs (A, ψ), where A is a chain algebra over R and $\psi : A \to A\otimes A$ is a map of chain algebras, while

$$\mathbf{wH}\big((A,\psi),(A',\psi')\big) = \{f \in \mathcal{A}\text{-}\mathbf{Alg}(A, A') \mid \psi' f = (f\otimes f)\psi\}.$$

Lemma 1.4. *The cobar construction extends to a functor* $\widetilde{\Omega} : \mathbf{wF} \to \mathbf{wH}$.

Proof. Given an object (C, Ψ) of **wF**, let $\widetilde{\Omega}(C,\Psi) = \big(\Omega C, q\,\mathrm{Ind}(\Psi)\big)$, where $\mathrm{Ind}(\Psi) : \Omega C \to \Omega(C\otimes C)$, as in Theorem 1.1(i), and $q : \Omega(C\otimes C) \to \Omega C\otimes \Omega C$ is Milgram's equivalence (0.1). In particular, $q\,\mathrm{Ind}(\Psi) : \Omega C \to \Omega C\otimes\Omega C$ is indeed a morphism of algebras, as it is a composite of two algebra maps.

On the other hand, given $\theta \in \mathbf{wF}\big((C,\Psi),(C',\Psi')\big)$, let $\widetilde{\Omega}\theta = \mathrm{Ind}(\theta) : \Omega C \to \Omega C'$. Then

$$\begin{aligned}\big(q\,\mathrm{Ind}(\Psi')\big)\widetilde{\Omega}\theta =& q\,\mathrm{Ind}(\Psi')\,\mathrm{Ind}(\theta) = q\,\mathrm{Ind}(\Psi'\theta)\\ =& q\,\mathrm{Ind}\big((\theta\wedge\theta)\Psi\big) = q\,\mathrm{Ind}(\theta\wedge\theta)\,\mathrm{Ind}(\Psi)\\ =& \big(\mathrm{Ind}(\theta)\otimes \mathrm{Ind}(\theta)\big)q\,\mathrm{Ind}(\Psi)\\ =& \big(\widetilde{\Omega}(\theta)\otimes\widetilde{\Omega}(\theta)\big)\big(q\,\mathrm{Ind}(\Psi)\big),\end{aligned}$$

i.e., $\widetilde{\Omega}\theta$ is indeed a morphism in **wH**. $\square$

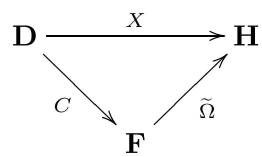
We are, of course, particularly interested in those objects (C, Ψ) of **wF** for which $\widetilde{\Omega}(C,\Psi)$ is actually a strict Hopf algebra, i.e., such that $q\,\mathrm{Ind}(\Psi)$ is coassociative.

Definition. The *Alexander-Whitney category* $\mathbf{F}$ is the full subcategory of $\mathbf{wF}$ such that (C,Ψ) is an object of $\mathbf{F}$ if and only if $q\,\mathrm{Ind}(\Psi)$ is coassociative. The objects of $\mathbf{F}$ are called *Alexander-Whitney coalgebras*.

As we explain in section 4, for any reduced simplicial set K, there is a canonical choice of Ψ_K such that $(C(K),\Psi_K)$ is an object of $\mathbf{F}$.

From the proof of Lemma 1.4, it is clear that $\widetilde{\Omega}$ restricts to a functor $\widetilde{\Omega}:\mathbf{F}\to\mathbf{H}$, where $\mathbf{H}$ is the category of Hopf algebras.

Theorem 1.5. *Let $X,Y:\mathbf{D}\to\mathbf{H}$ be functors, where $\mathbf{D}$ is a category admitting a set of models $\mathfrak{M}$ with respect to which Y is acyclic. Suppose that X factors through $\mathbf{F}$ as follows*



where C is free with respect to $\mathfrak{M}$. Let $\theta:UX\to UY$ be any natural transformation of functors into $\mathcal{A}$-$\mathbf{Alg}$, where $U:\mathbf{H}\to\mathcal{A}$-$\mathbf{Alg}$ denotes the forgetful functor. Then there exists a natural transformation $\widehat{\theta}:\Omega X\to\Omega Y$ extending the desuspension of θ, i.e., for all objects d in D,

$$\widehat{\theta}(d)=s^{-1}\theta(d)s+\textit{higher-order terms.}$$

The proof of this result depends strongly on the notion of a free functor with respect to a set of models, which we recall in detail, before commencing the proof of the theorem.

Let $\mathbf{D}$ be a category, and let $\mathfrak{M}$ be a set of objects in $\mathbf{D}$. A functor $X:\mathbf{D}\to\mathbf{M}$ is *free* with respect to $\mathfrak{M}$ if there is a set $\{e_M\in X(M)\mid M\in\mathfrak{M}\}$ such that $\{X(f)(e_M)\mid f\in\mathbf{D}(M,D),M\in\mathfrak{M}\}$ is an R-basis of $X(D)$ for all objects D in $\mathbf{D}$. If $X:\mathbf{D}\to\mathbf{H}$, then X is *free* with respect to $\mathfrak{M}$ if $U'X:\mathbf{D}\to\mathbf{M}$ is free, where $U':\mathbf{H}\to\mathbf{M}$ is the forgetful functor.

Proof. According to Theorem 1.1, it suffices to construct a natural transformation $\tau:\mathcal{T}(C)\underset{\mathcal{A}}{\circ}\mathcal{F}\to\mathcal{T}(Y)$ of right $\mathcal{A}$-modules such that $\tau(-\otimes z_0)=\theta$. We can then set $\widehat{\theta}=\mathrm{Ind}(\tau)$.

Since $\mathcal{T}(C)\underset{\mathcal{A}}{\circ}\mathcal{F}=\mathcal{T}(C)\circ\mathcal{S}\circ\mathcal{A}$, which is a free right $\mathcal{A}$-module, any natural tranformation of functors into the category of symmetric sequences $\mathcal{T}(C)\circ\mathcal{S}\to\mathcal{T}(Y)$ can be freely extended to a natural tranformation of functors into the category of right $\mathcal{A}$-modules. Furthermore, any family of equivariant natural transformations of functors into the category of *graded R-modules*

$$\{\tau_k:C\otimes\mathcal{S}(k)\to Y^{\otimes k}\mid k\geq 1\}\tag{1.2}$$

induces a natural tranformation of functors into symmetric sequences of *graded* *modules* $\tau' : \mathcal{T}(C) \circ \mathcal{S} \to \mathcal{T}(Y)$, given by composites like

$$\begin{array}{ccc} \mathcal{T}(C)(k) \otimes \mathcal{S}(n_1) \otimes \cdots \otimes \mathcal{S}(n_k) & \xrightarrow{\cong} & \big(C \otimes \mathcal{S}(n_1)\big) \otimes \cdots \otimes \big(C \otimes \mathcal{S}(n_k)\big) \\ & & \downarrow{\scriptstyle \tau_{n_1} \otimes \cdots \otimes \tau_{n_k}} \\ & & Y^{\otimes n_1} \otimes \cdots \otimes Y^{\otimes n_k} \\ & & \downarrow{\scriptstyle =} \\ & & Y^{\otimes n}, \end{array}$$

where $n = \sum_i n_i$. The free extension of τ' to $\tau : \mathcal{T}(C) \underset{\mathcal{A}}{\circ} \mathcal{F} \to \mathcal{T}(Y)$ will be a natural transformation of functors into $\mathbf{M}$, the category of chain complexes, if for all k,

$$\partial_{Y^{\otimes k}} \tau_k = \tau'(\partial_C \otimes 1) + \tau(1 \otimes \partial_{\mathcal{F}}) \tag{1.3}$$

where $\partial_{Y^{\otimes k}}$, ∂_C and $\partial_{\mathcal{F}}$ are the differentials on $Y^{\otimes k}$, C, and $\mathcal{F}$, respectively. Note that the formula for the restriction of τ' to $\big(\mathcal{T}(C) \circ \mathcal{S}\big)(k)$ involves only the τ_j's for $j \leq k$, as does the restriction of τ to $\big(\mathcal{T}(C) \circ \mathcal{S} \circ \mathcal{A}\big)(k)$.

We construct the family (1.2) recursively. We can choose τ_1 to be the "linear part" of θ, i.e., for all objects d in $\mathbf{D}$, the map of graded modules $\tau_1(d)(- \otimes z_0)$ is the composite

$$C(d) \xrightarrow{s^{-1}} s^{-1}C(d) \xrightarrow{\theta(d)|_{s^{-1}C(d)}} \Omega Y(d) \xrightarrow{\pi} s^{-1}Y(d) \xrightarrow{s} Y(d).$$

Assume that τ_k has been defined for all $k < n$ and that $\tau_n(e_M \otimes z_{n-1})$ has been defined for all M such that $\deg e_M < m$ so that (1.3) holds. Let M be an element of $\mathfrak{M}$ such that $\deg e_M = m$. According to the induction hypothesis, both $\tau'(\partial_{C(M)} e_M \otimes z_{n-1})$ and $\tau(e_M \otimes \partial_{\mathcal{F}} z_{n-1})$ have already been defined. Moreover, since $\partial_{C(M)} e_M \otimes z_{n-1} + (-1)^n e_M \otimes \partial_{\mathcal{F}} z_{n-1}$ is a cycle,

$$\tau'(\partial_{C(M)} e_M \otimes z_{n-1}) + (-1)^n \tau(e_M \otimes \partial_{\mathcal{F}} z_{n-1})$$

is a cycle in $Y(M)$ and therefore a boundary, since Y is acyclic with respect to M. We can thus continue the recursive construction of τ_n. $\square$

2. Homological perturbation theory

In this section we recall those elements of homological perturbation theory that we use in the construction of the Alexander-Whitney cobar diagonal.

Definition. Suppose that $\nabla : (X, \partial) \to (Y, d)$ and $f : (Y, d) \to (X, \partial)$ are morphisms of chain complexes. If $f\nabla = 1_X$ and there exists a chain homotopy $\varphi : (Y, d) \to (Y, d)$ such that

1. $d\varphi + \varphi d = \nabla f - 1_Y$,
2. $\varphi \nabla = 0$,
3. $f \varphi = 0$, and
4. $\varphi^2 = 0$,

then $(X,d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y,d) \circlearrowleft \varphi$ is a *strong deformation retract (SDR) of chain complexes.*

It is easy to show that given a chain homotopy φ' satisfying condition (1), there exists a chain homotopy φ satisfying all four conditions. As explained in, e.g., [LS], we can replace φ' by

$$\varphi = (\nabla f - 1_Y)\varphi'(\nabla f - 1_Y)d(\nabla f - 1_Y)\varphi'(\nabla f - 1_Y),$$

satisfying conditions (1)–(4).

When solving problems in homological or homotopical algebra, one often works with chain complexes with additional algebraic structure, e.g., chain algebras or coalgebras. It is natural to extend the notion of SDR's to categories of such objects.

Definition. An SDR $(X,d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y,d) \circlearrowleft \varphi$ is a *SDR of chain (co)algebras* if

(1) ∇ and f are morphisms of chain (co)algebras, and
(2) φ is a (co)derivation homotopy from ∇f to 1_Y.

The following notion, introduced by Gugenheim and Munkholm, is somewhat weaker than the previous definition for chain coalgebras but perhaps more useful.

Definition. An SDR $(X,d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y,d) \circlearrowleft \varphi$ is called *Eilenberg-Zilber (E-Z) data* if (Y,d,Δ_Y) and (X,d,Δ_X) are chain coalgebras and ∇ is a morphism of coalgebras.

Observe that in this case

$$(d\otimes 1_Y + 1_Y\otimes d)\big((f\otimes f)\Delta_Y\varphi\big) + \big((f\otimes f)\Delta_Y\varphi\big)d = \Delta_X f - (f\otimes f)\Delta_Y,$$

i.e., f is a map of coalgebras up to chain homotopy. In fact, f is a DCSH map, as Gugenheim and Munkholm showed in the following theorem [GM, Thm. 4.1], which proves extremely useful in section 4 of this article.

Theorem 2.1 [GM]. *Let $(X,d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y,d) \circlearrowleft \varphi$ be E-Z data such that Y is simply connected and X is connected. Let $F_1 = f$. Given F_i for all $i<k$, let*

$$F_k = -\sum_{i+j=k}(F_i\otimes F_j)\Delta_Y\varphi.$$

Similarly, let $\Phi_1 = \varphi$, and, given Φ_i for all $i<k$, let

$$\Phi_k = \big(\Phi_{k-1}\otimes 1_Y + \sum_{i+j=k}\nabla^{\otimes i}F_i\otimes\Phi_j\big)\Delta_Y\varphi.$$

Then

$$\Omega(X,d) \underset{\widetilde{\Omega}f}{\overset{\Omega\nabla}{\rightleftarrows}} \Omega(Y,d) \circlearrowleft \widetilde{\Omega}\varphi$$

is an SDR of chain algebras, where $\widetilde{\Omega}f = \sum_{k\geq 1}(s^{-1})^{\otimes k}F_k s$ and $\widetilde{\Omega}\varphi = \sum_{k\geq 1}(s^{-1})^{\otimes k}\Phi_k s$.

Let **EZ** be the category with as objects E-Z data $(X,d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y,d) \circlearrowleft \varphi$ such that Y is simply connected and X is connected. A morphism in **EZ**

$$\Big((X,d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y,d) \circlearrowleft \varphi\Big) \longrightarrow \Big((X',d') \underset{f'}{\overset{\nabla'}{\rightleftarrows}} (Y',d') \circlearrowleft \varphi'\Big)$$

consists of a pair of morphisms of chain coalgebras $g : (X,d) \longrightarrow (X',d')$ and $h : (Y,d) \longrightarrow (Y',d')$ such that $h\nabla = \nabla' g$, $gf = f'h$ and $h\varphi = \varphi' h$.

Corollary 2.2. *There is a functor* $AW : \mathbf{EZ} \to \big((\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}\big)^{\to}$.

Proof. Set $AW\big((X,d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y,d) \circlearrowleft \varphi\big)$ equal to $\operatorname{Ind}^{-1}(\widetilde{\Omega} f) \in (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}(Y,X)$, where $\widetilde{\Omega} f$ is defined as in Theorem 2.1. The evident naturality of the definition of $\widetilde{\Omega} f$ implies that AW is a functor. $\square$

We call AW the *Alexander-Whitney functor*.

Fundamental Example. The natural Eilenberg-Zilber and Alexander-Whitney equivalences for simplicial sets provide the most classic example of E-Z data and play a crucial role in the constructions in this article. Let K and L be two simplicial sets. Define morphisms on their normalized chain complexes

$$\nabla_{K,L} : C(K) \otimes C(L) \to C(K \times L) \quad \text{and} \quad f_{K,L} : C(K \times L) \to C(K) \otimes C(L)$$

by

$$\nabla_{K,L}(x \otimes y) = \sum_{(\mu,\nu) \in \mathcal{S}_{p,q}} (-1)^{\operatorname{sgn}(\mu)} (s_{\nu_q} ... s_{\nu_1} x, s_{\mu_p} ... s_{\mu_1} y)$$

where $\mathcal{S}_{p,q}$ denotes the set of (p,q)-shuffles, $\operatorname{sgn}(\mu)$ is the signature of μ and $x \in K_p$, $y \in L_q$, and

$$f_{K,L}((x,y)) = \sum_{i=0}^{n} \partial_{i+1} \cdots \partial_n x \otimes \partial_0^i y$$

where $(x,y) \in (K \times L)_n$. We call $\nabla_{K,L}$ the *shuffle (or Eilenberg-Zilber) map* and $f_{K,L}$ the *Alexander-Whitney map*. There is a chain homotopy, $\varphi_{K,L}$, so that

$$C(K) \otimes C(L) \underset{f_{K,L}}{\overset{\nabla_{K,L}}{\rightleftarrows}} C(K \times L) \circlearrowleft \varphi_{K,L}, \tag{2.1}$$

is an SDR of chain complexes. Furthermore $\nabla_{K,L}$ is a map of coalgebras, with respect to the usual coproducts, which are defined in terms of the natural equivalence $f_{K,L}$. We have thus defined a functor

$$EZ : \mathbf{sSet_1} \times \mathbf{sSet_1} \to \mathbf{EZ},$$

where $\mathbf{sSet_1}$ is the category of 1-reduced simplicial sets. We call EZ the *Eilenberg-Zilber functor*.

When K and L are 1-reduced, we can apply Theorem 2.1 to the SDR (2.1) and obtain a new SDR

$$\Omega\big(C(K) \otimes C(L)\big) \underset{\widetilde{\Omega} f_{K,L}}{\overset{\Omega\nabla_{K,L}}{\rightleftarrows}} \Omega C(K \times L) \circlearrowleft \widetilde{\Omega}\varphi_{K,L}. \tag{2.2}$$

In the language of Corollary 2.2, we have a functor from the category $\mathbf{sSet_1} \times \mathbf{sSet_1}$ to $\big((\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}\big)^{\to}$, given by the following composite.

$$\mathbf{sSet_1} \times \mathbf{sSet_1} \xrightarrow{EZ} \mathbf{EZ} \xrightarrow{AW} \big((\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}\big)^{\to}$$

See May's book [M, §28] and the articles of Eilenberg and MacLane [EM1], [EM2] for further details.

We can apply our knowledge of this fundamental example to proving the following important result.

Theorem 2.3. *Let* $\mathbf{sSet}_1$ *denote the category of* 1*-reduced simplicial sets. There is a functor* $\widetilde{C} : \mathbf{sSet}_1 \longrightarrow \mathbf{wF}$ *such that* $U\widetilde{C} = C$*, the normalized chains functor, where* $U : \mathbf{wF} \longrightarrow \mathcal{A}$*-*$\mathbf{Coalg}$ *is the forgetful functor.*

Proof. Given a 1-reduced simplicial set K, observe that

$$(AW \circ EZ)(K,K) \in (\mathcal{A}, \psi_{\mathcal{F}})\text{-}\mathbf{Coalg}\big(C(K \times K), C(K) \otimes C(K)\big).$$

Define Ψ_K to be the composite

$$\mathcal{T}\big(C(K)\big) \underset{\mathcal{A}}{\diamond} \mathcal{F} \xrightarrow{\mathcal{T}\left((\Delta_K)_\sharp\right) \underset{\mathcal{A}}{\diamond} 1} \mathcal{T}\big(C(K \times K)\big) \underset{\mathcal{A}}{\diamond} \mathcal{F} \xrightarrow{AW \circ EZ(K,K)} \mathcal{T}\big(C(K) \otimes C(K)\big).$$

The pair $\big(C(K), \Psi_K\big)$ is a weak Alexander-Whitney coalgebra, so we can set

$$\widetilde{C}(K) := \big(C(K), \Psi_K\big).$$

It is then immediate that $U\widetilde{C}(K) = C(K)$.

Given a morphism $h : K \longrightarrow L$ of 1-reduced simplicial sets, let $\widetilde{C}(h) \in \mathbf{F}\big(\widetilde{C}(K), \widetilde{C}(L)\big)$ be the morphism of right $\mathcal{A}$-modules

$$\mathcal{T}(h_\sharp) \underset{\mathcal{A}}{\diamond} \varepsilon : \mathcal{T}\big(C(K)\big) \underset{\mathcal{A}}{\diamond} \mathcal{F} \to \mathcal{T}\big(C(L)\big),$$

where $\varepsilon : \mathcal{F} \longrightarrow \mathcal{A}$ is the co-unit of $\mathcal{F}$. A straightforward diagram chase enables us to establish that $\Psi_L \widetilde{C}(h) = \big(\widetilde{C}(h) \wedge \widetilde{C}(h)\big)\Psi_K$, ensuring that $\widetilde{C}(h)$ really is a morphism in $\mathbf{F}$. Key to the success of the diagram chase are the naturality of AW and EZ and of the diagonal map on simplicial sets, as well as the fact that $(\varepsilon \underset{\mathcal{A}}{\diamond} 1)\psi_{\mathcal{F}} = Id_{\mathcal{F}} = (1 \underset{\mathcal{A}}{\diamond} \varepsilon)\psi_{\mathcal{F}}$, i.e., that ε is a counit for $\psi_{\mathcal{F}}$. □

3. Twisting cochains and twisting functions

We recall here the algebraic notion of a twisting cochain and the simplicial notion of a twisting function, both of which are crucial in this article. We explain the relationship between the two, which is expressed in terms of a perturbation of the Eilenberg-Zilber SDR defined in section 2. We conclude by recalling an important result of Morace and Prouté [MP] concerning the relationship between Szczarba's twisting cochain and the Eilenberg-Zilber equivalence

Definition. Let (C,d) be a chain coalgebra with coproduct Δ, and let (A,d) be a chain algebra with product μ. A *twisting cochain* from (C,d) to (A,d) is a degree -1 map $t : C \longrightarrow A$ of graded modules such that

$$dt + td = \mu(t \otimes t)\Delta.$$

The definition of a twisting cochain $t : C \longrightarrow A$ is formulated precisely so that the following two constructions work smoothly. First, let $(A,d) \otimes_t (C,d) = (A \otimes C, D_t)$, where $D_t = d \otimes 1_C + 1_A \otimes d - (\mu \otimes 1_C)(1_A \otimes t \otimes 1_C)(1_A \otimes \Delta)$. It is easy to see that $D_t^2 = 0$, so that $(A,d) \otimes_t (C,d)$ is a chain complex, which extends (A,d).

Second, if C is connected, let $\theta : Ts^{-1}C_+ \longrightarrow A$ be the algebra map given by $\theta(s^{-1}c) = t(c)$. Then θ is in fact a chain algebra map $\theta : \Omega(C,d) \longrightarrow (A,d)$, and the complex $(A,d)\otimes_t (C,d)$ is acyclic if and only if θ is a quasi-isomorphism. It is equally clear that any algebra map $\theta : \Omega(C,d) \longrightarrow (A,d)$ gives rise to a twisting cochain via the composition

$$C_+ \xrightarrow{s^{-1}} s^{-1}C_+ \hookrightarrow Ts^{-1}C_+ \xrightarrow{\theta} A.$$

In particular, for any two chain coalgebras (C,d,Δ) and (C',d',Δ'), the set of DCSH maps from C to C' and the set of twisting cochains from C to $\Omega C'$ are naturally in bijective correspondence.

The twisting cochain associated to the cobar construction is a fundamental example of this notion. Let (C,d,Δ) be a simply-connected chain coalgebra. Consider the linear map

$$t_{\Omega C} : C \longrightarrow \Omega C : c \longrightarrow s^{-1}c.$$

It is a easy exercise to show that $t_{\Omega C}$ is a twisting cochain and induces the identity map on ΩC. Thus, in particular, $(\Omega C,d)\otimes_{t_\Omega} (C,d)$ is acyclic; this is the well-known acyclic cobar construction [HMS].

Definition/Lemma. Let $t : C \longrightarrow A$ and $t' : C' \longrightarrow A'$ be twisting cochains. Let $\varepsilon : C \longrightarrow \mathbb{Z}$ and $\varepsilon' : C' \longrightarrow \mathbb{Z}$ be counits, and let $\eta : \mathbb{Z} \longrightarrow A$ and $\eta' : \mathbb{Z} \longrightarrow A'$ be units. Set

$$t * t' = t \otimes \eta'\varepsilon' + \eta\varepsilon \otimes t' : C \otimes C' \longrightarrow A \otimes A'$$

Then $t * t'$ is a twisting cochain, called the *cartesian product* of t and t'. If $\theta : \Omega C \longrightarrow A$ and $\theta' : \Omega C' \longrightarrow A'$ are the chain algebra maps induced by t and t', then we write $\theta * \theta' : \Omega(C \otimes C') \longrightarrow A \otimes A'$ for the chain algebra map induced by $t * t'$.

Remark. Observe that Milgram's equivalence $q : \Omega(C\otimes C') \longrightarrow \Omega C \otimes \Omega C'$ is exactly $Id_{\Omega C} * Id_{\Omega C'}$, which is the chain algebra map induced by $t_{\Omega C} * t_{\Omega C'}$.

Definition. Let K be a simplicial set and G a simplicial group, where the neutral element in any dimension is noted e. A degree -1 map of graded sets $\tau : K \longrightarrow G$ is a *twisting function* if

$$\begin{aligned}
\partial_0\tau(x) &= \big(\tau(\partial_0 x)\big)^{-1}\tau(\partial_1 x)\\
\partial_i\tau(x) &= \tau(\partial_{i+1}x) \quad \text{for all } i>0\\
s_i\tau(x) &= \tau(s_{i+1}x) \quad \text{for all } i\geq 0\\
\tau(s_0x) &= e
\end{aligned}$$

for all $x \in K$.

The definition of a twisting function $\tau : K \longrightarrow G$ is formulated precisely so that if G operates on the left on a simplicial set L, then we can construct a *twisted cartesian product* of K and L, denoted $L \times_\tau K$, which is a simplicial set such that $(L\times_\tau K)_n = L_n \times K_n$, with faces and degeneracies given by

$$\begin{aligned}
\partial_0(y,x) &= (\tau(x)\cdot\partial_0 y, \partial_0 x)\\
\partial_i(y,x) &= (\partial_i y, \partial_i x) \quad \text{for all } i>0\\
s_i(y,x) &= (s_i y, s_i x) \quad \text{for all } i\geq 0.
\end{aligned}$$

If L is a Kan complex, then the projection $L\times_\tau K \longrightarrow K$ is a Kan fibration [M].

Example. The canonical twisting functions $\lambda_K : K \to GK : x \mapsto \bar{x}$ are particularly important in this article, in particular because the geometric realization of $GK \times_{\lambda_K} K$ is acyclic.

Twisting cochains and twisting functions are, not surprisingly, very closely related. The theorem below describes their relationship in terms of a generalization of the Eilenberg-Zilber/Alexander-Whitney equivalences.

Theorem 3.1. *For each twisting function $\tau : K \to G$ there exists a twisting cochain $t(\tau) : C(K) \to C(G)$ and an SDR*

$$C(G) \otimes_{t(\tau)} C(K) \underset{f_\tau}{\overset{\nabla_\tau}{\rightleftarrows}} C(G \times_\tau K) \circlearrowleft \varphi_\tau .$$

Furthermore the choice of $t(\tau)$, ∇_τ, f_τ and φ_τ can be made naturally.

Observe that since the realization of $GK \times_{\lambda_K} K$ is acyclic, $C(GK) \otimes_{t(\lambda_K)} C(K)$ is acyclic as well, for any natural choice of twisting cochain $t(-)$ fulfilling the conditions of the theorem above. Consequently, the induced chain algebra map $\theta(\lambda_K) : \Omega C(K) \to C(GK)$ is a quasi-isomorphism.

E. Brown proved the original version of this theorem, for topological spaces, by methods of acyclic models [B]. Somewhat later R. Brown [Br] and Gugenheim [G] used homological perturbation theory to prove the existence of $t(\tau)$ in the simplicial case without defining it explicitly. Szczarba was the first to give an explicit, though extremely complex, formula for $t(\tau)$, in [Sz].

Convention. Henceforth in this article, the notation $sz(\tau)$ will be used exclusively to mean Szczarba's explicit twisting cochain, while sz_K will always denote $sz(\lambda_K)$ and Sz_K the chain algebra map induced by sz_K.

Recently, in [MP] Morace and Prouté provided an alternate, more compact construction of $sz(\tau)$, which enabled them to prove that sz_K commutes with the shuffle map, as described below.

Theorem 3.2 [MP]. *Let K and L be reduced simplicial sets. Let $\rho : G(K \times L) \to GK \times GL$ denote the homomorphism of simplicial groups defined in the introduction. Then the diagram of graded module maps*

$$\begin{array}{ccc}
C(K) \otimes C(L) & \xrightarrow{\nabla_{K,L}} & C(K \times L) \\
 & & \big\downarrow {\scriptstyle sz_{K \times L}} \\
\big\downarrow {\scriptstyle sz_K * sz_L} & & C(G(K \times L)) \\
 & & \big\downarrow {\scriptstyle \rho_\sharp} \\
C(GK) \otimes C(GL) & \xrightarrow{\nabla_{GK,GL}} & C(GK \times GL)
\end{array}$$

commutes.

The following corollary of Theorem 3.2 is crucial to the development in the next section.

Corollary 3.3. *Let K and L be 1-reduced simplicial sets, and let ρ be as above. Then the diagram of chain algebra maps*

$$\begin{array}{ccc}
\Omega C(K\times L) & \xrightarrow{\widetilde{\Omega} f_{K,L}} & \Omega\big(C(K)\otimes C(L)\big) \\
\downarrow{\scriptstyle Sz_{K\times L}} & & \\
C(G(K\times L)) & & \downarrow{\scriptstyle Sz_K * Sz_L} \\
\downarrow{\scriptstyle \rho_\sharp} & & \\
C(GK\times GL) & \xrightarrow{f_{GK,GL}} & C(GK)\otimes C(GL)
\end{array}$$

commutes up to homotopy of chain algebras.

Proof. Recall that ∇ is always a map of coalgebras, so that it induces a map of chain algebras $\Omega\nabla$ on cobar constructions. As an immediate consequence of Theorem 3.2, we obtain that the diagram of chain algebras

$$\begin{array}{ccc}
\Omega\big(C(K)\otimes C(L)\big) & \xrightarrow{\Omega\nabla_{K,L}} & \Omega C(K\times L) \\
 & & \downarrow{\scriptstyle Sz_{K\times L}} \\
\downarrow{\scriptstyle Sz_K * Sz_L} & & C(G(K\times L)) \\
 & & \downarrow{\scriptstyle \rho_\sharp} \\
C(GK)\otimes C(GL) & \xrightarrow{\nabla_{GK,GL}} & C(GK\times GL)
\end{array}$$

commutes. It suffices to check the commutativity for generators of $\Omega\big(C(K)\otimes C(L)\big)$, i.e, for elements of $s^{-1}\big(C(K)\otimes C(L)\big)_+$, which is equivalent to the commutativity of the diagram in Theorem 3.2.

If $\Phi = f_{GK,GL}\circ\rho_\sharp\circ Sz_{K\times L}\circ\widetilde{\Omega}\varphi_{K,L} : \Omega C(K\times L)\to C(GK)\otimes C(GL)$, then

$$\begin{aligned}
(d\otimes 1+1\otimes d)\Phi+\Phi d &= f_{GK,GL}\circ\rho_\sharp\circ Sz_{K\times L}\circ\Omega\nabla\circ\widetilde{\Omega} f_{K,L} - f_{GK,GL}\circ\rho_\sharp\circ Sz_{K\times L}\\
&= f_{GK,GL}\circ\nabla_{GK,GL}\circ Sz_K * Sz_L\circ\widetilde{\Omega} f_{K,L} - f_{GK,GL}\circ\rho_\sharp\circ Sz_{K\times L}\\
&= Sz_K * Sz_L\circ\widetilde{\Omega} f_{K,L} - f_{GK,GL}\circ\rho_\sharp\circ Sz_{K\times L}.
\end{aligned}$$

The map Φ is thus a chain homotopy from $Sz_K * Sz_L\circ\widetilde{\Omega} f_{K,L}$ to $f_{GK,GL}\circ\rho_\sharp\circ Sz_{K\times L}$. Furthermore, since, according to Theorem 2.1, $\widetilde{\Omega}\varphi$ is a $(\Omega\nabla\circ\widetilde{\Omega} f, 1)$-derivation, Φ is a $(Sz_K * Sz_L\circ\widetilde{\Omega} f_{K,L}, f_{GK,GL}\circ\rho_\sharp\circ Sz_{K\times L})$-derivation. The diagram in the statement of the theorem commutes therefore up to homotopy of chain algebras. □

4. The canonical Adams-Hilton model

Our goal in this section is to define and establish the key properties of the Alexander-Whitney cobar diagonal. Throughout the section we abuse notation slightly and write f_K, ∇_K and φ_K instead of $f_{K,K}$, $\nabla_{K,K}$ and $\varphi_{K,K}$. Recall furthermore the functors $\widetilde{\Omega} : \mathbf{wF}\to\mathbf{wH}$ (Lemma 1.4) and $\widetilde{C} : \mathbf{sSet}_1\to\mathbf{wF}$ (Theorem 2.3).

Definition. Let K be a 1-reduced simplicial set. The *canonical Adams-Hilton model* for K is $\widetilde{\Omega}\widetilde{C}(K)$. The coproduct ψ_K on the canonical Adams-Hilton model is called the *Alexander-Whitney (A-W) cobar diagonal*.

Unrolling the definition of ψ_K, we see that it is equal to the following composite.

$$\Omega C(K) \xrightarrow{\Omega(\Delta_K)_\sharp} \Omega C(K\times K) \xrightarrow{\widetilde{\Omega} f_K} \Omega(C(K)\otimes C(K)) \xrightarrow{q} \Omega C(K)\otimes \Omega C(K).$$

We show in the next two results that it is, in particular, cocommutative up to homotopy of chain algebras and strictly coassociative. Thus, $\widetilde{\Omega}\widetilde{C}(K)\in \mathbf{H}$, i.e., $\widetilde{C}(K)$ is a *strict* Alexander-Whitney coalgebra.

Proposition 4.1. *The Alexander-Whitney cobar diagonal ψ_K is cocommutative up to homotopy of chain algebras for all 1-reduced simplicial sets K.*

Proof. Consider the following diagram, in which sw denotes both the simplicial coordinate switch map and the algebraic tensor switch map.

$$\begin{array}{ccccccc}
\Omega C(K) & \xrightarrow{\Omega(\Delta)_\sharp} & \Omega C(K\times K) & \xrightarrow{\widetilde{\Omega} f} & \Omega(C(K)\otimes C(K)) & \xrightarrow{q} & \Omega C(K)\otimes\Omega C(K)\\
 & {\searrow}^{\Omega(\Delta)_\sharp} & \downarrow{\scriptstyle \Omega(sw)_\sharp} & & \downarrow{\scriptstyle \Omega(sw)} & & \downarrow{\scriptstyle sw}\\
 & & \Omega C(K\times K) & \xrightarrow{\widetilde{\Omega} f} & \Omega(C(K)\otimes C(K)) & \xrightarrow{q} & \Omega C(K)\otimes\Omega C(K)
\end{array}$$

The triangle on the left and the square on the right commute for obvious reasons, while the middle square commutes up to chain homotopy, as

$$\begin{aligned}
\Omega(sw)\circ\widetilde{\Omega} f &= \widetilde{\Omega} f\circ\Omega\nabla\circ\Omega(sw)\circ\widetilde{\Omega} f\\
&= \widetilde{\Omega} f\circ\Omega(sw)_\sharp\circ\Omega\nabla\circ\widetilde{\Omega} f \qquad \text{since } \nabla\circ sw=(sw)_\sharp\circ\nabla\\
&\simeq \widetilde{\Omega} f\circ\Omega(sw)_\sharp.
\end{aligned}$$

The homotopy in the last step is provided by $\widetilde{\Omega} f\circ\Omega(sw)_\sharp\circ\widetilde{\Omega}\varphi$. Hence, the whole diagram commutes up to chain homotopy, where $q\circ\widetilde{\Omega} f\circ\Omega(sw)_\sharp\circ\widetilde{\Omega}\varphi\circ\Omega\Delta_\sharp$ provides the necessary homotopy. $\square$

Theorem 4.2. *The Alexander-Whitney cobar diagonal ψ_K is strictly coassociative for all 1-reduced simplicial sets K.*

Proof. We need to show that $(\psi_K\otimes 1)\psi_K=(1\otimes\psi_K)\psi_K$, which means that we need to show that the following diagram commutes. (Note that we drop the subscript K for the remainder of this proof.)

$$\begin{array}{ccccccc}
\Omega C(K) & \xrightarrow{\Omega(\Delta)_\sharp} & \Omega C(K^2) & \xrightarrow{\widetilde{\Omega} f} & \Omega(C(K)^{\otimes 2}) & \xrightarrow{q} & (\Omega C(K))^{\otimes 2}\\
\downarrow{\scriptstyle \Omega(\Delta)_\sharp} & & & & & & \downarrow{\scriptstyle 1\otimes\Omega(\Delta)_\sharp}\\
\Omega C(K^2) & & & & & & \Omega C(K)\otimes\Omega C(K^2)\\
\downarrow{\scriptstyle \widetilde{\Omega} f} & & & & & & \downarrow{\scriptstyle 1\otimes\widetilde{\Omega} f}\\
\Omega(C(K)^{\otimes 2}) & & & & & & \Omega C(K)\otimes(\Omega C(K)^{\otimes 2})\\
\downarrow{\scriptstyle q} & & & & & & \downarrow{\scriptstyle 1\otimes q}\\
(\Omega C(K))^{\otimes 2} & \xrightarrow{\Omega(\Delta)_\sharp\otimes 1} & \Omega C(K^2)\otimes\Omega C(K) & \xrightarrow{\widetilde{\Omega} f\otimes 1} & \Omega(C(K)^{\otimes 2})\otimes\Omega C(K) & \xrightarrow{q\otimes 1} & (\Omega C(K))^{\otimes 3}
\end{array}$$

In order to prove that the square above commutes, we divide it into nine smaller squares

$$\begin{array}{ccccccc}
\Omega C(K) & \xrightarrow{\Omega(\Delta)_\sharp} & \Omega C(K^2) & \xrightarrow{\widetilde{\Omega} f} & \Omega(C(K)^{\otimes 2}) & \xrightarrow{q} & (\Omega C(K))^{\otimes 2} \\
\downarrow{\scriptstyle \Omega(\Delta)_\sharp} & & \downarrow{\scriptstyle \Omega(1\times\Delta)_\sharp} & & \downarrow{\scriptstyle \Omega(1\otimes(\Delta)_\sharp)} & & \downarrow{\scriptstyle 1\otimes\Omega(\Delta)_\sharp} \\
\Omega C(K^2) & \xrightarrow{\Omega(\Delta\times 1)_\sharp} & \Omega C(K^3) & \xrightarrow{\widetilde{\Omega} f_{K,K^2}} & \Omega(C(K)\otimes C(K^2)) & \xrightarrow{q} & \Omega C(K)\otimes \Omega C(K^2) \\
\downarrow{\scriptstyle \widetilde{\Omega} f} & & \downarrow{\scriptstyle \widetilde{\Omega} f_{K^2,K}} & & \downarrow{\scriptstyle \widetilde{\Omega}(1\otimes f)} & & \downarrow{\scriptstyle 1\otimes\widetilde{\Omega} f} \\
\Omega(C(K)^{\otimes 2}) & \xrightarrow{\Omega((\Delta)_\sharp\otimes 1)} & \Omega(C(K^2)\otimes C(K)) & \xrightarrow{\widetilde{\Omega}(f\otimes 1)} & \Omega(C(K)^{\otimes 3}) & \xrightarrow{q} & \Omega C(K)\otimes(\Omega C(K)^{\otimes 2}) \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle 1\otimes q} \\
(\Omega C(K))^{\otimes 2} & \xrightarrow{\Omega(\Delta)_\sharp\otimes 1} & \Omega C(K^2)\otimes\Omega C(K) & \xrightarrow{\widetilde{\Omega} f\otimes 1} & \Omega(C(K)^{\otimes 2})\otimes\Omega C(K) & \xrightarrow{q\otimes 1} & (\Omega C(K))^{\otimes 3}
\end{array}$$

and show that each of the small squares commutes, with one exception, for which we can correct. We label each small square with its row and column number, so that, e.g., square $(2,3)$ is

$$\begin{array}{ccc}
\Omega(C(K)\otimes C(K^2)) & \xrightarrow{q} & \Omega C(K)\otimes\Omega C(K^2) \\
\downarrow{\scriptstyle \widetilde{\Omega}(1\otimes f)} & & \downarrow{\scriptstyle 1\otimes\widetilde{\Omega} f} \\
\Omega(C(K)^{\otimes 3}) & \xrightarrow{q} & \Omega C(K)\otimes(\Omega C(K)^{\otimes 2}).
\end{array}$$

The commutativity of eight of the nine small squares is immediate. Square $(1,1)$ commutes since Δ is coassociative. Squares $(1,2)$ and $(2,1)$ commute by naturality of f, while squares $(1,3)$ and $(3,1)$ commute by naturality of q. The commutativity of squares $(2,3)$ and $(3,2)$ is an immediate consequence of Proposition 2.2. Finally, a simple calculation shows that square $(3,3)$ commutes as well.

Let $q^{(2)} = (1\otimes q)q = (q\otimes 1)q$. In the case of square $(2,2)$, we show that

$$Im\big(\widetilde{\Omega}(1\otimes f)\circ\widetilde{\Omega} f_{K,K^2} - \widetilde{\Omega}(f\otimes 1)\circ\widetilde{\Omega} f_{K^2,K}\big) \subseteq \ker q^{(2)},$$

which suffices to conclude that the large square commutes, since we know that the other eight small squares commute.

Let $c_{1,2}$ and $c_{2,1}$ denote the usual coproducts on $C(K)\otimes C(K^2)$ and $C(K^2)\otimes C(K)$, respectively. Given any $z\in C(K^3)$, use the Einstein summation convention in writing $f_{K,K^2}(z) = x_i\otimes y^i$, $c_K(x_i) = x_{i,j}\otimes x_i^j$, and $c_{K^2}(\varphi(y^i)) = \varphi(y^i)_k\otimes\varphi(y^i)^k$, so that

$$\begin{aligned}
&q^{(2)}\big(s^{-1}(1\otimes f)\big)^{\otimes 2} c_{1,2}(1\otimes\varphi) f_{K,K^2}(z) \\
&\qquad = q^{(2)}\big(s^{-1}(1\otimes f)\big)^{\otimes 2} c_{1,2}\big(x_i\otimes\varphi(y^i)\big) \\
&\qquad = q^{(2)}\big(s^{-1}(1\otimes f)\big)^{\otimes 2}\big(\pm x_{i,j}\otimes\varphi(y^i)_k\otimes x_i^j\otimes\varphi(y^i)^k\big) \\
&\qquad = q^{(2)}\Big(\pm s^{-1}\big(x_{i,j}\otimes f(\varphi(y^i)_k)\big)s^{-1}\big(x_i^j\otimes f(\varphi(y^i)^k)\big)\Big) \\
&\qquad = (1\otimes q)\big((s^{-1}x_i\otimes 1)(1\otimes s^{-1}f\varphi(y^i)) \pm (1\otimes s^{-1}f\varphi(y^i))(s^{-1}x_i\otimes 1)\big)
\end{aligned}$$

since $(1\otimes q)(s^{-1}(u\otimes v))=0$ unless $|u|=0$ or $|v|=0$. This last sum is 0, however, since $f\varphi=0$.

Similarly, $q^{(2)}\big(s^{-1}(f\otimes 1)\big)^{\otimes 2}c_{2,1}(\varphi\otimes 1)f_{K^2,K}(z)=0$. Applying Gugenheim and Munkholm's formula from Theorem 2.1, we obtain for all $z\in C(K^3)$

$$\begin{aligned} q^{(2)}\widetilde{\Omega}(1\otimes f)\widetilde{\Omega}f_{K,K^2}(s^{-1}z) &= q^{(2)}s^{-1}(1\otimes f)f_{K,K^2}(z)\\ &= q^{(2)}s^{-1}(f\otimes 1)f_{K^2,K}(z)\\ &= q^{(2)}\widetilde{\Omega}(f\otimes 1)\widetilde{\Omega}f_{K^2,K}(s^{-1}z), \end{aligned}$$

since in general

$$(1\otimes f_{L,M})f_{K,L\times M}=(f_{K,L}\otimes 1)f_{K\times L,M}:C(K\times L\times M)\to C(K)\otimes C(L)\otimes C(M). \quad \square$$

Proposition 4.3. *The chain algebra quasi-isomorphism $Sz_K:\Omega C(K)\to C(GK)$ induced by Szczarba's twisting cochain sz_K is a map of chain coalgebras up to homotopy of chain algebras, with respect to the Alexander-Whitney cobar diagonal and the usual coproduct $c_{GK}=f_{GK}\circ(\Delta_{GK})_\sharp$ on $C(GK)$, i.e., the diagram*

$$\begin{array}{ccc} \Omega C(K) & \xrightarrow{\psi_K} & \Omega C(K)\otimes\Omega C(K) \\ \downarrow{\scriptstyle Sz_K} & & \downarrow{\scriptstyle Sz_K\otimes Sz_K} \\ C(GK) & \xrightarrow{c_{GK}} & C(GK)\otimes C(GK) \end{array}$$

commutes up to homotopy of chain algebras.

Remark. Since c_{GK} is homotopy cocommutative, Proposition 4.3 implies immediately that ψ_K is homotopy cocommutative as well. We consider, however, that it is worthwhile to establish the homotopy cocommutativity of ψ_K independently, as we do in Proposition 4.1, since we obtain an explicit formula for the chain homotopy.

Proof. We can expand and complete the diagram in the statement of the theorem to obtain the diagram below.

$$\begin{array}{cccc} \Omega C(K) & \xrightarrow{Sz_K} & C(GK) & \\ \downarrow{\scriptstyle \Omega(\Delta_K)_\sharp} & & \downarrow{\scriptstyle (G\Delta_K)_\sharp} & \\ \Omega C(K\times K) & \xrightarrow{Sz_{K^2}} & C\big(G(K\times K)\big) & (\Delta_{GK})_\sharp \\ \downarrow{\scriptstyle \widetilde{\Omega}f_K} & & \downarrow{\scriptstyle \rho_\sharp} & \\ \Omega(CK\otimes CK) & & C(GK\times GK) & \\ \downarrow{\scriptstyle q} & \searrow{\scriptstyle Sz_K * Sz_K} & \downarrow{\scriptstyle f_{GK}} & \\ \Omega CK\otimes\Omega CK & \xrightarrow{Sz_K\otimes Sz_K} & C(GK)\otimes C(GK) & \end{array}$$

The top square commutes exactly, by naturality of the twisting cochains. An easy calculation shows that the bottom triangle commutes exactly. Since Corollary 3.3

implies that the middle square commutes up to homotopy of chain algebras, we can conclude that the theorem is true. In particular $f_{GK}\rho_\sharp Sz_{K\times K}\widetilde{\Omega}\varphi_K\Omega(\Delta_K)_\sharp$ is an appropriate derivation homotopy. □

It would be interesting to determine under what conditions Sz_K is a strict map of Hopf algebras. We have checked that $(Sz_K \otimes Sz_K)\psi_K = c_{GK}Sz_K$ up through degree 3 and will show in a later paper [HPS2] that Sz_K is a strict Hopf algebra map when K is a suspension.

Even if Sz_K is not a strict coalgebra map, we know that it is at least the next best thing, as stated in the following theorem.

Theorem 4.4. *Any natural map $\theta_K : \Omega C(K) \to C(GK)$ of chain algebras is a DCSH map, with respect to any natural choice of strictly coassociative coproduct χ_K on $\Omega C(K)$.*

Remark. Proposition 4.3 is, of course, an immediate corollary of Theorem 4.4. The independent proof of Proposition 4.3 serves to provide an explicit formula for the homotopy between $Sz_K \otimes Sz_K)\psi_K$ and $c_{GK}Sz_K$. The proof below sacrifices all hope of explicit formulae on the altar of extreme generality.

Proof. Let $\overline{\Delta}[n]$ denote the quotient of the standard simplicial n-simplex $\Delta[n]$ by its 0-skeleton. Recall from [MP] that there is a contracting chain homotopy $\bar{h} : C(G\overline{\Delta}[n]) \to C(G\overline{\Delta}[n])$. The functor $C\big(G(-)\big)$ from reduced simplicial sets to connected chain algebras is therefore acyclic on the set of models $\mathfrak{M} = \{\overline{\Delta}[n] \mid n \geq 0\}$.

On the other hand, the functor C from reduced simplicial sets to connected chain coalgebras is free on $\mathfrak{M}$. In particular, the set $\{\iota_n \in C(\overline{\Delta}[n]) \mid n \geq 0\}$ gives rise to basis of $C(K)$ for all K, where ι_n denotes the unique nondegenerate n-simplex of $\overline{\Delta}[n]$.

Theorem 4.4 therefore follows immediately from Theorem 1.5, where $\mathbf{D}$ is the category of reduced simplicial sets, $X = (\Omega C(-), \chi_-)$ and $Y = C\big(G(-)\big)$. □

Remark. The results in this section beg the question of iteration of the cobar construction. Let K be any 2-reduced simplicial set. Since ψ_K is strictly coassociative and Sz_K is a DCSH map, we can apply the cobar construction to the quasi-isomorphism $Sz_K : \widetilde{\Omega}\widetilde{C}(K) = (\Omega C(K), \psi_K) \to C(GK)$ and consider the composite

$$\Omega\widetilde{\Omega}\widetilde{C}(K) \xrightarrow{\widetilde{\Omega}Sz_K} \Omega C(GK) \xrightarrow{Sz_{GK}} C(G^2K),$$

which is a quasi-isomorphism of chain algebras (see Lemma 1.4 and Theorem 2.3 for explanation of the notation $\widetilde{\Omega}$ and $\widetilde{C}$). The question is now whether there is a canonical, topologically-meaningful way to define a coassociative coproduct on $\Omega\widetilde{\Omega}\widetilde{C}(K)$, in order to iterate the process. In other words, is there a natural, coassociative coproduct on $\Omega\widetilde{\Omega}\widetilde{C}(K)$ with respect to which $Sz_{GK}\circ\widetilde{\Omega}Sz_K$ is a DCSH map? Equivalently, does $\widetilde{\Omega}\widetilde{C}(K)$ admit a natural Alexander-Whitney coalgebra structure, with respect to which Sz_K is a morphism in $\mathbf{F}$?

Using the notion of the *diffraction* from [HPS] and the more general version of the Cobar Duality Theorem proved there, we can show that $\widetilde{\Omega}\widetilde{C}(K)$ admits a natural *weak* Alexander-Whitney structure, with respect to which Sz_K is a morphism in $\mathbf{wF}$. Consequently, $\Omega\widetilde{\Omega}\widetilde{C}(K)$ indeed admits a natural coproduct, but it is not necessarily coassociative, which prevents us from applying the cobar construction again.

In [HPS2] we show that if EK is the suspension of a 1-reduced simplicial set K, then $\widetilde{\Omega}\widetilde{C}(EK)$ does admit a natural, strict Alexander-Whitney coalgebra structure and that Sz_{EK} is then a morphism in $\mathbf{F}$. We conjecture that this result generalizes to higher suspensions and correspondingly higher iterations of the cobar construction.

5. The Baues coproduct and the Alexander-Whitney cobar diagonal

We show in this section that the coproduct defined by Baues in [Ba] is the same as the Alexander-Whitney cobar diagonal defined in section 4. As mentioned in the introduction, this result is at first sight quite surprising, since there is an obvious asymmetry in Baues's combinatorial definition, which is well hidden in our definition of the A-W cobar diagonal.

We begin by recalling the definition of Baues's coproduct on $\Omega C(K)$, where K is a 1-reduced simplicial set. For any $m \leq n \in \mathbb{N}$, let $[m,n] = \{j \in \mathbb{N} \mid m \leq j \leq n\}$. Let Δ denote the category with objects

$$Ob\Delta = \{[0,n] \mid n \geq 0\}$$

and

$$\Delta\big([0,m],[0,n]\big) = \{f : [0,m] \longrightarrow [0,n] | f \text{ order-preserving set map}\}.$$

Viewing the simplicial set K as a contravariant functor from Δ to the category of sets, given $x \in K_n := K([0,n])$ and $0 \leq a_1 < a_2 < \cdots < a_m \leq n$, let

$$x_{a_1 \ldots a_m} := K(\mathbf{a})(x) \in K_m$$

where $\mathbf{a} : [0,m] \longrightarrow [0,n] : j \mapsto a_j$.

Let x be a nondegenerate n-simplex of K. Baues's coproduct $\widetilde{\psi}$ on $\Omega C(K)$ is defined by

$$\widetilde{\psi}(s^{-1}x) = \sum_{\substack{0 \leq m < n \\ 0 < a_1 < \cdots < a_m < n}} (-1)^{\ell(\mathbf{a})} s^{-1}x_{0\ldots a_1} s^{-1}x_{a_1 \ldots a_2} \cdots s^{-1}x_{a_m \ldots n} \otimes s^{-1}x_{0a_1 \ldots a_m n}$$

where

$$\ell(\mathbf{a}) = (a_1 - 1) + \Big(\sum_{i=2}^{m} (i-1)(a_i - a_{i-1} - 1)\Big) + m(n - a_m - 1).$$

Baues showed in [Ba] that $\widetilde{\psi}$ was strictly coassociative and that it was cocommutative up to derivation homotopy, providing an explicit derivation homotopy for the cocommutativity.

To prove that the A-W cobar diagonal agrees with the Baues coproduct, we examine closely each summand of $\widetilde{\Omega} f_{K,L}$, determining precisely what survives upon composition with $q : \Omega(CK \otimes CL) \longrightarrow \Omega CK \otimes \Omega CL$. Henceforth, in the interest of simplifying the notation, we no longer make signs explicit.

Recall from section 2 the Eilenberg–Zilber SDR of normalized chain complexes

$$C(K)\otimes C(L) \underset{f_{K,L}}{\overset{\nabla_{K,L}}{\rightleftarrows}} C(K\times L) \circlearrowleft \varphi_{K,L},$$

where we can rewrite the definitions of $f_{K,L}$ and $\nabla_{K,L}$ in terms the notation introduced above as follows. In degree n,

$$(\nabla_{K,L})_n(x\otimes y) = \sum_{\ell=0}^{n} \sum_{\substack{A\cup B=[0,n-1]\\ |A|=n-\ell, |B|=\ell}} \pm(s_A x, s_B y) \tag{5.1}$$

and

$$(f_{K,L})_n(x,y) = \sum_{\ell=0}^{n} x_{0\ldots\ell}\otimes y_{\ell\ldots n} \tag{5.2}$$

where s_I denotes $s_{i_r}\cdots s_{i_2}s_{i_1}$ for any set I of non-negative integers $i_1<i_2<\cdots<i_r$ and $|I|$ denotes the cardinality of I. There is also a recursive formula for $\varphi_{K,L}$, due to Eilenberg and MacLane [EM2]. Let $g=\nabla_{K,L}f_{K,L}$. Then

$$(\varphi_{K,L})_n = -(g)'s_0 + \big((\varphi_{K,L})_{n-1}\big)', \tag{5.3}$$

where the prime denotes the derivation operation on simplicial operators, i.e.,

$$h = s_{j_n}\cdots s_{j_0}\partial_{i_0}\cdots\partial_{i_m} \Rightarrow h' = s_{j_n+1}\cdots s_{j_0+1}\partial_{i_0+1}\cdots\partial_{i_m+1}.$$

Let $\widehat{\varphi}$ denote the degree $+1$ map

$$\widehat{\varphi}: C(K\times L)\xrightarrow{\varphi_{K,L}} C(K\times L)\xrightarrow{(\Delta_{K\times L})_\sharp} C((K\times L)^2)\xrightarrow{f_{K\times L,K\times L}} C(K\times L)^{\otimes 2}.$$

If K and L are 0-reduced, consider the pushout $CK\vee CL$ of the complexes CK and CL over $\mathbb{Z}$ and the map of chain complexes

$$\kappa: CK\otimes CL\longrightarrow CK\vee CL$$

defined by $\kappa(x\otimes 1)=x$, $\kappa(1\otimes y)=y$ and $\kappa(x\otimes y)=0$ if $|x|,|y|>0$. Define a family of linear maps

$$\overline{\mathcal{F}} = \{\overline{F}_k : C(K\times L)\longrightarrow (CK\vee CL)^{\otimes k} \mid \deg \overline{F} = k-1, k\geq 1\}$$

by

$$\overline{F}_1 : C(K\times L)\xrightarrow{f_{K,L}} CK\otimes CL \xrightarrow{\kappa} CK\vee CL \tag{5.4}$$

$$\overline{F}_k : C(K\times L)\xrightarrow{\widehat{\varphi}} C(K\times L)^{\otimes 2}\xrightarrow{-\sum_{i+j=k}\overline{F}_i\otimes\overline{F}_j} (CK\vee CL)^{\otimes k}. \tag{5.5}$$

We can use the family $\overline{\mathcal{F}}$ to obtain a useful factorization of $q\circ\widetilde{\Omega}f_{K,L}$, as follows.

Observe first, by comparison with the construction given after the statement of Theorem 2.1, that

$$\bar{F}_k = \kappa^{\otimes k} F_k \tag{5.6}$$

where F_k is defined as in Theorem 2.1. Next note that for any pair of 1-connected chain coalgebras (C, d) and (C', d'), the algebra map $\gamma : Ts^{-1}(C_+ \vee C'_+) \longrightarrow Ts^{-1}C_+ \otimes Ts^{-1}C'_+$ specified by $\gamma(s^{-1}c) = s^{-1}c \otimes 1$ and $\gamma(s^{-1}c') = 1 \otimes s^{-1}c'$ for $c \in C$ and $c' \in C'$ commutes with the cobar differentials, i.e., it is a chain algebra map $\gamma : \Omega((C, d) \vee (C', d')) \longrightarrow \Omega(C, d) \otimes \Omega(C', d')$. Furthermore,

$$q = \gamma \circ T(s^{-1}\kappa s) : \Omega(CK \otimes CL) \longrightarrow \Omega CK \otimes \Omega CL,$$

which implies, by (5.6), that when K and L are 1-reduced,

$$q\widetilde{\Omega} f_{K,L} = \gamma \circ \sum_{k\geq 1} (s^{-1})^{\otimes k} \bar{F}_k s : \Omega C(K \times L) \longrightarrow \Omega C(K) \otimes C(L).$$

In particular, the A-W cobar diagonal is equal to $\gamma \circ \sum_{k\geq 1}(s^{-1})^{\otimes k}\bar{F}_k s \circ \Omega(\Delta_K)_\sharp$.

Thanks to the decomposition above, we obtain as an immediate consequence of the next theorem that

$$\psi_K = \widetilde{\psi}. \tag{5.7}$$

Theorem 5.1. *Let* $\overline{\mathcal{F}} = \{\overline{F}_k : C(K \times L) \longrightarrow (CK \vee CL)^{\otimes k} \mid \deg \overline{F} = k-1, k \geq 1\}$ *denote the family defined above. If* $j \geq 2$ *then*

$$(\overline{F}_i \otimes \overline{F}_j)\,\widehat{\varphi} = 0 \tag{5.8}$$

If K *and* L *are 1-reduced then*

$$\overline{F}_k(x, y) = \sum_{\{0<i_1<\cdots<i_r<n\}} \pm y_{0i_1\ldots i_r n} \otimes x_{0\ldots i_1} \otimes \cdots \otimes x_{i_r\ldots n} \tag{5.9}$$

In this summation we adopt the convention that 1-simplices x_J *or* y_J *for* $|J| = 2$ *are to be identified to the unit (not to zero), and consider only those terms for which exactly* k *non-trivial tensor factors remain.*

To see how Theorem 5.1 implies (5.7), note that (5.9) implies that for all $x \in C_n K$,

$$\gamma\Big(\sum_{k\geq 1}(s^{-1})^{\otimes k}\overline{F}_k(x, x)\Big) = \sum_{\{0<i_1<\cdots<i_r<n\}} \pm s^{-1}x_{0\ldots i_1}\cdots s^{-1}x_{i_r\ldots n} \otimes s^{-1}x_{0i_1\ldots i_r n}.$$

In the proof of Theorem 5.1 we rely on the following lemmas.

The map $\overline{F}_1$ is easy to identify, by definition of f and κ.

Lemma 5.2. *For all (x,y) in $K \times L$, we have $\overline{F}_1(x,y) = x + y$.*

To understand $\overline{F}_k$ for $k \geq 2$, we use the following explicit formula for φ.

Lemma 5.3. *Let A, B be disjoint sets such that $A \cup B = [m+1, n]$ and $|B| = r - m$ for $r > m$. For $(x,y) \in K_n \times L_n$, let*

$$\varphi^{A,B}(x,y) = (s_{A\cup\{m\}}\, x_{0\ldots r},\, s_B\, y_{0\ldots mr\ldots n}) \in K_{n+1} \times L_{n+1}.$$

Then the Eilenberg–Zilber homotopy φ is given by

$$\varphi(x,y) = \sum_{\substack{m<r \\ A\cup B=[m+1,n] \\ |A|=n-r,\ |B|=r-m}} \pm\varphi^{A,B}(x,y).$$

The following result should be compared with Lemma 5.2 and the classical result $\varphi^2 = 0$.

Lemma 5.4. *For a general term $\varphi^{A,B}(x,y)$ of Lemma 5.3 and $0 \leq \ell \leq n+1$,*

$$\overline{F}_1((\varphi^{A,B}(x,y))_{0\ldots\ell}) = \begin{cases} x_{0\ldots\ell} \ + \ y_{0\ldots\ell} & : \ell \leq m, \\ y_{0\ldots mr\ldots r-m+\ell-1} & : \ell > m \text{ and } [m+1,\ell-1] \subseteq A, \\ 0 & : \text{else} \end{cases}$$

$$\overline{F}_1((\varphi^{A,B}(x,y))_{\ell\ldots n+1}) = \begin{cases} x_{r-n+\ell-1\ldots r} & : \ell > m \text{ and } [\ell,n] \subseteq B, \\ y_{\ell-1\ldots n} & : \ell > m \text{ and } [\ell,n] \subseteq A, \\ 0 & : \text{else.} \end{cases}$$

$$\varphi((\varphi^{A,B}(x,y))_{0\ldots\ell}) = \quad 0 \qquad : \ell > m \text{ and } [m+1,\ell-1] \not\subseteq A$$

$$\varphi((\varphi^{A,B}(x,y))_{\ell\ldots n+1}) = \quad 0 \qquad \text{always.}$$

Proof of Theorem 5.1. If $j \geq 2$, then we have

$$\overline{F}_j((\varphi^{A,B}(x,y))_{\ell\ldots n+1}) \ = \ -\sum_{j_1+j_2=j} (\overline{F}_{j_1} \otimes \overline{F}_{j_2})\, f\, \Delta_\sharp\, \varphi((\varphi^{A,B}(x,y))_{\ell\ldots n+1}) \ = \ 0$$

by the final result of Lemma 5.4, and so

$$(\overline{F}_i \otimes \overline{F}_j)\widehat{\varphi}(x,y) \ = \sum_{\ell,m,r,A,B} \overline{F}_i((\varphi^{A,B}(x,y))_{0\ldots\ell}) \otimes \overline{F}_j((\varphi^{A,B}(x,y))_{\ell\ldots n+1}) \ = \ 0,$$

proving the first part of the theorem.

For the second part, note that for $k = 1$ the right hand side of (5.9) reduces to

$$y_{0n} \otimes x_{0\ldots n} \ + \ y_{0\ldots n} \otimes x_{01} \otimes \cdots \otimes x_{n-1n}$$

which we identify with $x + y = \overline{F}_1(x,y)$.

For $k = 2$ we use the first two results of Lemma 5.4 and the fact that $B \neq \emptyset$ so A cannot contain both $[m+1,\ell-1]$ and $[\ell,n]$, to establish that

$$\begin{aligned} \overline{F}_2(x,y) &= \sum_{\ell,m,r,A,B} \overline{F}_1((\varphi^{A,B}(x,y))_{0\ldots\ell}) \otimes \overline{F}_1((\varphi^{A,B}(x,y))_{\ell\ldots n+1}) \\ &= \sum_{\ell,m,r} \ y_{0\ldots mr\ldots r-m+\ell-1} \otimes x_{r-n+\ell-1\ldots r}\,. \end{aligned}$$

Here $[m+1,\ell-1]=A$ and $[\ell,n]=B$, so $r-m=|B|=n-\ell+1$ and we have

$$\overline{F}_2(x,y) \;=\; \sum_{m<r} y_{0\ldots mr\ldots n} \;\otimes\; x_{m\ldots r}$$

which agrees with (5.9). For $k\geq 2$ we have, by (5.8),

$$\overline{F}_{k+1}(x,y) \;=\; \sum_{\substack{\ell,m,r,A,B\\ i+j=k}} \big((\overline{F}_i\otimes\overline{F}_j)\,f\Delta_\sharp\,\varphi((\varphi^{A,B}(x,y))_{0\ldots\ell})\big)\otimes\overline{F}_1((\varphi^{A,B}(x,y))_{\ell\ldots n+1})$$

and Lemma 5.4 tells us once again we must take $A=[m+1,\ell-1]$, $B=[\ell,n]$, $\ell=n+m-r+1$ to obtain non-vanishing terms

$$\overline{F}_{k+1}(x,y) \;=\; \sum_{m<r} \overline{F}_k(s_{[m,m+n-r]}\,x_{0\ldots m},\, y_{0\ldots mr\ldots n}) \;\otimes\; x_{m\ldots r}$$

The theorem then follows by a straightforward induction. □

Proof of Lemma 5.3. Expanding the definitions (5.1–5.3) we get

$$\begin{aligned}
\varphi_n(x,y) &= \sum_{m=0}^{n-1} \pm g^{(m+1)}_{n-m}s_m\,(x,y)\\
&= \sum_{\substack{0\leq m\leq n-1\\ 0\leq \ell\leq n-m\\ A\cup B=[0,n-m-1]\\ |A|=n-m-\ell,|B|=\ell}} \pm(s_{A+m+1}\partial^{n-m-\ell}_{\ell+1+m+1}s_m\,x, s_{B+m+1}\partial^{\ell}_{0+m+1}s_m\,y)\\
&= \sum_{\substack{0\leq m\leq n-1\\ 1\leq \ell\leq n-m\\ A\cup B=[m+1,n]\\ |A|=n-m-\ell,|B|=\ell}} \pm(s_As_m\partial^{n-m-\ell}_{m+\ell+1}\,x, s_B\partial^{\ell-1}_{m+1}\,y)\\
&= \sum_{\substack{0\leq m\leq n-1\\ m+1\leq r\leq n\\ A\cup B=[m+1,n]\\ |A|=n-r,|B|=r-m}} \pm(s_{A\cup\{m\}}\,x_{0\ldots r},\, s_B\,y_{0\ldots mr\ldots n}).
\end{aligned}$$

Here we have dropped the $\ell=0$ terms, since they are degenerate (in the image of s_m), and written $r=m+\ell$. □

Proof of Lemma 5.4. By Lemma 5.2 the first equation holds for $\ell\leq m$, since then

$$\overline{F}_1((\varphi^{A,B}(x,y))_{0\ldots\ell}) \;=\; \overline{F}_1((x,y)_{0\ldots\ell}) \;=\; x_{0\ldots\ell}\;+\;y_{0\ldots\ell}$$

If $\ell>m$, then the first term is always s_m-degenerate, as is the second, unless all the indices specified in B are $\geq\ell$, that is, unless $[m+1,\ell-1]\subseteq A$.

If $\ell\leq m$, then $(\varphi^{A,B}(x,y))_{\ell\ldots n+1}$ always has an s_m-degeneracy in the first component and some other degeneracy in the second, since $B\neq\emptyset$, so its image under $\overline{F}_1$ is zero. If $\ell>m$, then the first component can only be non-degenerate if no elements of A are $\geq\ell$, so $[\ell,n]\subseteq B$. Similarly the second can be non-degenerate only if $[\ell,n]\subseteq A$.

If $\ell > m$ then every term of $\varphi((\varphi^{A,B}(x,y))_{0,\dots,\ell})$ has as first factor the face of the second component specified by $[0,p] \cup [q,\ell]$, say, which is degenerate if $q \leq m$ and $[q,\ell-1] \cap B$ is non-empty. If $q > m$, however, then either $p > m$ and $[p,q] \subseteq B$, so it is still degenerate, or $[p,q] \cap (A \cup \{m\})$ is non-empty and the other factor is degenerate.

A similar argument shows that the terms of $\varphi((\varphi^{A,B}(x,y))_{\ell\dots n+1})$ always have one of the two factors degenerate. □

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