# On the Design of Economic NMPC Based on an Exact Turnpike Property

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**Abstract:** We discuss the design of sampled-data economic nonlinear model predictive control schemes for continuous-time systems. We present novel sufficient convergence conditions that do not require any kind of terminal constraints nor terminal penalties. Instead, the proposed convergence conditions are based on an exact turnpike property of the underlying optimal control problem. We prove that, in the presence of state constraints, the existence of an exact turnpike implies recursive feasibility of the optimization. We draw upon the example of optimal fish harvest to illustrate our findings.

*Keywords:* economic model predictive control, stability, turnpike property, optimal control

## 1. INTRODUCTION

Recently, there has been a widespread interest in nonlinear model predictive control (NMPC) schemes that are not tailored to stabilization around a setpoint but rather to optimization of transient performance. In Rawlings and Amrit (2009), the name *economic MPC* is coined for these approaches. Note that very similar ideas have been discussed previously in the context of process control under the label *dynamic real-time optimization* by Kadam and Marquardt (2007).

A recent overview article by Ellis et al. (2014) points out that turnpike properties are an intrinsic feature of optimal control problems (OCP) arising in economic NMPC. The term *turnpike* describes a property of OCPs, whereby, for varying initial conditions and horizons, the computed solutions stay close to a specific steady state during the major part of the time horizon. The paper by Ellis et al. (2014) also mentions that only a few works—such as Grüne (2013); Rawlings and Amrit (2009); Würth et al. (2009)—deal explicitly with turnpike properties in the context of economic NMPC. This gap in the literature on NMPC is surprising, since turnpike properties are known to play an important role in the analysis of infinite-horizon OCPs, which frequently arise in optimal control approaches to economic systems, see Carlson et al. (1991); McKenzie (1976). The goal of the present paper is to partially close this gap by showing how the turnpike property of OCPs allows establishing sufficient convergence/stability [1] conditions for NMPC.

Often, the convergence/stability of NMPC is enforced via terminal constraints or terminal penalties that are added to the OCP that is solved at each sampling instant, see Mayne et al. (2000). Since, in general, terminal constraints tend to increase the computational burden of solving the OCP, several works have established convergence/stability via controllability assumptions to avoid these constraints, see Grüne and Pannek (2011); Jadbabaie and Hauser (2005). The main contribution of this paper is a turnpike-based approach addressing sufficient convergence conditions of sampled-data NMPC schemes without the addition of terminal constraints nor terminal penalties.

We show that an exact turnpike property allows establishing (i) finite-time convergence of sampled-data NMPC to the optimal steady state and (ii) recursive feasibility of the underlying optimal control problems. The proposed conditions do not require any specific structure of the cost function, such as lower boundedness of the distance to a setpoint by a class $\mathcal{K}$ function. Our approach uses techniques similar to those proposed in Grüne (2013). However, there are two main differences to the work of Grüne: While the former considers discrete-time economic NMPC, we consider the sampled-data case for continuous-time systems; and while the former proves stability based on a dissipativity assumption (that implies the presence of turnpike behavior in the OCP), we directly use a specific turnpike property to establish finite-time convergence. It is worth mentioning that the case of sampled-data economic NMPC for continuous-time systems has received much less attention than its discrete-time counterpart. One of the few works on this topic uses restrictive terminal constraints, see Alessandretti et al. (2014). Hence, the present paper seems to be the first work that establishes convergence of sampled-data economic NMPC for continuous-time systems and relies on turnpike properties instead of terminal constraints and/or terminal penalties.

The remainder of the paper is structured as follows. Section 2 describes a general sampled-data NMPC scheme. The notion of exact turnpike of OCPs and its properties are discussed in Section 3. The main NMPC stability result is presented in Section 4. A fish harvest problem is considered as an example in Section 5.

[1] Note that for discrete-time systems one usually shows stability of NMPC-controlled systems. In contrast, the majority of results on sampled-data NMPC for continuous-time systems merely establishes convergence via Barbalat's Lemma, cf. Fontes (2001).

## 2. SAMPLED-DATA NONLINEAR MODEL PREDICTIVE CONTROL

We consider the nonlinear plant given by

$$\dot{x}_p = f(x_p, u_p), \quad x_p(0) = x_0 \in \mathcal{X}_0, \tag{1}$$

where the state $x_p \in \mathbb{R}^{n_x}$ and the input $u_p \in \mathbb{R}^{n_u}$ are constrained to lie in the compact sets $\mathcal{X} \subset \mathbb{R}^{n_x}$ and $\mathcal{U} \subset \mathbb{R}^{n_u}$. The initial condition $x_0$ is constrained to the compact set $\mathcal{X}_0 \subseteq \mathcal{X}$. We assume that $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is Lipschitz on $\mathcal{X} \times \mathcal{U}$ and sufficiently often continuously differentiable.

We are interested in controlling the plant (1) by means of a sampled-data NMPC scheme similar to Findeisen et al. (2007); Fontes (2001). The NMPC scheme is based on receding-horizon solutions to an OCP. Hence, at each sampling instant $t_k = k\delta, k \in \mathbb{N}$, we propose to minimize the objective functional

$$J_T(x_p(t_k), u(\cdot)) = \frac{1}{T} \int_0^T F(x(\tau), u(\tau)) d\tau, \tag{2}$$

where $F : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is the cost function, $T \in \mathbb{R}^+$ is the prediction horizon, and $\delta > 0$ is the sampling time. We distinguish between the plant variables in (1) and the values predicted by the model by denoting the former with the subscript $(\cdot)_p$.

The NMPC scheme is based on receding-horizon solutions to the following OCP, denoted as $\mathcal{OCP}_T(x_p(t_k))$,

$$\underset{u(\cdot) \in \mathcal{M}([0,T],\mathcal{U})}{\text{minimize}} \; J_T(x_p(t_k), u(\cdot)) \tag{3a}$$

subject to

$$\forall \tau \in [0,T] : \; \frac{dx(\tau)}{d\tau} = f(x(\tau), u(\tau)), \; x(0) = x_p(t_k), \tag{3b}$$

$$u(\tau) \in \mathcal{U}, \; x(\tau) \in \mathcal{X}, \tag{3c}$$

where $\mathcal{M}([0,T],\mathcal{U})$ denotes the class of measurable functions on $[0,T]$ taking values in the compact set $\mathcal{U}$.

The purpose of the subsequent developments is to establish novel conditions ensuring that NMPC based on (3) leads to asymptotic convergence to a specific steady state. To this end, we make the following assumptions.

*Assumption 1.* The prediction model (3b) is identical to the plant (1), i.e., there is no plant-model mismatch.

*Assumption 2.* For any $x_0 \in \mathcal{X}$ and any input $u(\cdot) \in \mathcal{M}([0,\infty),\mathcal{U})$, plant (1) has a unique absolutely continuous solution.

Let $x(\cdot, x_p(t_k), u(\cdot))$ denote a solution to (3b) that starts at $x_p(t_k)$ at time $\tau = 0$ and is driven by the input $u(\cdot)$. The pair $(x(\cdot, x_p(t_k), u(\cdot)), u(\cdot))^T$ is said to be admissible if $u(\cdot) \in \mathcal{M}([0,T],\mathcal{U})$ and, for all $\tau \in [0,T]$, $x(\tau, x_p(t_k), u(\cdot)) \in \mathcal{X}$. An optimal solution to (3) is denoted as $u^\star(\cdot)$ and the corresponding state trajectory $x^\star(\cdot, x_p(t_k), u^\star(\cdot))$. [2] At the sampling instant $t_k$, the first part of the optimal solution $u^\star(\cdot, x_p(t_k))$ is applied i.e.

$$u_p(t_k + \tau) = u^\star(\tau, x_p(t_k)), \quad \forall \tau \in [0, \delta). \tag{4}$$

Notational remarks. We denote the dependence of (3) on the initial conditions $x_p(t_k)$ and the horizon length $T$ arising from the receding-horizon control strategy by writing $\mathcal{OCP}_T(x_p(t_k))$. While the time variable of the plant (1) is $t \geq 0$, the time variable of $\mathcal{OCP}_T(x_p(t_k))$ is denoted by $\tau \in [0,T]$. Admissible *pairs* are abbreviated by $z(\cdot) := (x(\cdot), u(\cdot))^T$. Occasionally, we want to highlight the dependence of an admissible pair or an admissible input on the initial condition $x_p(t_k)$, for which we write $z(\cdot, x_p(t_k)) := (x(\cdot, x_p(t_k), u(\cdot)), u(\cdot))^T$ and $u(\cdot, x_p(t_k))$. Likewise, we write $F(z(\cdot, x_p(t_k))) := F\left((x(\cdot, x_p(t_k), u(\cdot)), u(\cdot)\right)$. Steady-state values are indicated by the superscript $(\bar{\cdot})$, and thus we denote steady-state pairs by $\bar{z} := (\bar{x}, \bar{u})^T$.

[2] Here, we assume for simplicity that the optimal solution exists and can be attained. We refer to Lee and Markus (1967) for conditions ensuring the existence of optimal solutions to OCP (3).

## 3. EXACT TURNPIKE PROPERTIES OF OCPS

This paper investigates sufficient conditions for the convergence of plant (1) subject to the sampled-data NMPC scheme based on $\mathcal{OCP}_T(x_p(t_k))$. These conditions rely on *turnpike properties* that describe features of solutions to an OCP for varying initial conditions and horizon length. To this end, we consider in this section $\mathcal{OCP}_T(x_0)$ with $x_0 \in \mathcal{X}_0$ and $T > 0$.

### *Turnpike Properties*

*Definition 1.* (Input-state turnpike property).
The optimal solution pairs $z^\star(\cdot, x_0)$ of $\mathcal{OCP}_T(x_0)$ are said to have an *input-state turnpike property* with respect to the steady-state pair $\bar{z} = (\bar{x}, \bar{u})^T \in \mathcal{Z}$ if there exists a function $\nu : [0,\infty) \to [0,\infty)$ such that, for all $x_0 \in \mathcal{X}_0$ and all $T > 0$, we have

$$\mu[\Theta_{\varepsilon,T}] < \nu(\varepsilon) < \infty \quad \forall\, \varepsilon > 0, \tag{5}$$

where $\mu[\cdot]$ is the Lebesgue measure on the real line and

$$\Theta_{\varepsilon,T} := \{\tau \in [0,T] : \|z^\star(\tau, x_0) - \bar{z}\| > \varepsilon\}. \tag{6}$$

The pairs $z^\star(\cdot, x_0)$ of (3) are said to have an *exact input-state turnpike property* if Condition (5) also holds for $\varepsilon \to 0$, i.e.,

$$\lim_{\varepsilon \to 0} \mu[\Theta_{\varepsilon,T}] < \nu(0) < \infty. \tag{7}$$

The term turnpike property was coined by Dorfman et al. (1958). The turnpike property states that—for any initial condition $x_0 \in \mathcal{X}_0$ and any horizon length $T > 0$—the time that the optimal solutions spend outside an $\varepsilon$-neighborhood of $\bar{z}$ is bounded by $\nu(\varepsilon)$, where $\nu(\varepsilon)$ *is not a function of the horizon length* $T$. In essence, the turnpike property states the existence of an arc along which the optimal pair $z^\star(\cdot)$ stays close to the steady-state pair $\bar{z}$ in the sense of the Euclidean norm $\|\cdot\|$, and the length of this arc increases with increasing horizon length $T$. The exact turnpike property (7) requires that, for a sufficiently long horizon $T$, the optimal solutions have to be exactly at $\bar{z}$ for all $\tau \in [0,T] \backslash \Theta_{0,T}$. Note that if a turnpike property is not exact we call it *approximate.* [3]

Definition 1 is a variant of that used in Faulwasser et al. (2014), where turnpike solutions are required to be close only to the steady state $\bar{x}$, which may be denoted as a *state turnpike.* In contrast, we require here that the turnpike

[3] In Faulwasser et al. (2014) we used a slightly different terminology, denoting general turnpikes as being approximate. Herein, in order to be more precise, we adjust the terminology.

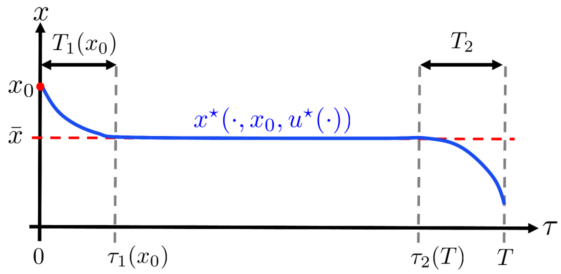



Fig. 1. Sketch of an exact turnpike.

solutions are close to the steady-state values of the state and the input, which we denote as an *input-state turnpike.*

A definition of the turnpike property based on exponential bounds on the trajectories is used in Damm et al. (2014) for discrete-time problems. For continuous-time systems a definition similar to (5) is implicitly given in Carlson et al. (1991).

*Remark 1.* (Dependence on $T$ and $x_0$).
According to Definition 1, the steady-state pair $\bar{z}$ at which the turnpike takes place has to be the same for all horizon lengths $T > 0$ and all initial conditions $x_0 \in \mathcal{X}_0$.

*Remark 2.* (Reachability of $\bar{x}$ and optimality of $\bar{z}$).
Definition 1 implies that the steady state $\bar{x}$, at which the approximate turnpike occurs, is asymptotically reachable from all $x_0 \in \mathcal{X}_0$. Furthermore, using ideas from Faulwasser et al. (2014), it can be shown that, under a reachability condition, $\bar{z} = (\bar{x}, \bar{u})^T$ is guaranteed to be an optimal solution to

$$\underset{(\bar{x},\bar{u})\,\in\,\mathbb{R}^{n_x}\times\mathbb{R}^{n_u}}{\text{minimize}} \; F(\bar{x}, \bar{u}) \tag{8a}$$

$$\text{subject to}$$

$$f(\bar{x}, \bar{u}) = 0 \tag{8b}$$

$$(\bar{x}, \bar{u})^T \in \mathcal{X} \times \mathcal{U}. \tag{8c}$$

In principle, the measure-based turnpike definition used here allows for pathological cases, in which the optimal solutions pass the optimal steady state infinitely often within some time interval such that the measure of $\Theta_{0,T}$ is larger than zero. Hence, we make the following assumption to exclude such pathological cases.

*Assumption 3.* (Non-pathological exact turnpike). For all $x_0 \in \mathcal{X}_0$ and a sufficiently long horizon $T \geq T_{min}$, $\mathcal{OCP}_T(x_0)$ has a *non-pathological exact* input-state turnpike at $\bar{z} = (\bar{x}, \bar{u})^T$ such that there exist non-negative constants $T_1(x_0)$ and $T_2$ which, with $\tau_1(x_0) = T_1(x_0)$ and $\tau_2(T) = T - T_2$, leads to $0 \leq \tau_1(x_0) < \tau_2(T) \leq T$. Furthermore, the optimal pairs $z^\star(\cdot, x_0)$ satisfy $z^\star(\tau, x_0) = \bar{z}$ for all $\tau \in [\tau_1(x_0), \tau_2(T)]$ and $z^\star(\tau, x_0) \neq \bar{z}$ for all $\tau \notin [\tau_1(x_0), \tau_2(T)]$.

A graphical interpretation of this assumption is sketched in Figure 1. In essence, it is required that, for a sufficiently long horizon, the optimal solutions enter the turnpike exactly at time $\tau_1(x_0) = T_1(x_0)$ and leave the turnpike at time $\tau_2(T) = T - T_2$.[4] Note that the duration $T_1(x_0)$ of the turnpike-approaching part of the optimal solutions, in general, depends on the initial condition $x_0$. We will show later that the duration $T_2$ of the turnpike-leaving part of the optimal solutions is independent of $x_0$ and $T$.[5] Several examples that satisfy Assumption 3 can be found in the literature, cf. Fuller's problem (Zelikin and Borisov (1994)), the optimal fish harvest problem discussed in Cliff and Vincent (1973), the protein folding problem presented in Coron et al. (2014) and the ressource allocation problems shown in Bryson and Ho (1969); Clarke (2013).

*Remark 3.* (Solutions may stay close to the turnpike).
Definition 1 allows optimal solutions to leave the neighborhood of $\bar{z}$ at the end of the optimization horizon as illustrated in Figure 1. However, this is not required. In other words, solutions that enter a neighborhood of $\bar{z}$ and do not leave this neighborhood later are also called turnpike solutions.

*Remark 4.* (Turnpikes, parametric OCPs and NMPC).
We want to stress here that turnpike properties are properties of parametric OCPs, i.e., they describe common features of the solution trajectories under variation of the initial conditions and the horizon length. Since NMPC typically relies on the receding-horizon solution to the same OCP for varying initial conditions, turnpike properties are natural candidates to help establish convergence conditions for NMPC.

*Properties of Exact Turnpike Solutions*

Next, we investigate several helpful technical properties that characterize exact turnpike solutions.

*Lemma 1.* (Identical end pieces).
*Consider $\mathcal{OCP}_T(x_i)$ with $x_i \in \{x_1, x_2\}$ and let Assumption 3 hold. Then, for any $x_1, x_2 \in \mathcal{X}_0$ and a sufficiently long horizon $T \geq T_{min}$, the optimal pairs $z_1^\star(\cdot, x_1)$ and $z_2^\star(\cdot, x_2)$ satisfy*

$$\frac{1}{T}\int_{\tilde{\tau}}^{T} F(z_1^\star(\tau, x_1))d\tau = \frac{1}{T}\int_{\tilde{\tau}}^{T} F(z_2^\star(\tau, x_2))d\tau \tag{9}$$

*for all $\tilde{\tau} \in [\max\{\tau_1(x_1), \tau_1(x_2)\}, T]$.*

**Proof.** Without loss of generality, let us assume a sufficiently large $T$ such that, at time $\tilde{\tau}$, we have $z^\star(\tilde{\tau}, x_i, u^\star(\cdot, x_i)) = \bar{z}$, $x_i \in \{x_1, x_2\}$, i.e., the optimal pairs of $\mathcal{OCP}_T(x_1)$ and $\mathcal{OCP}_T(x_2)$ have reached the turnpike.

Let $\mathcal{OCP}_{T-\tilde{\tau}}(x^\star(\tilde{\tau}, x_i, u^\star(\cdot)))$ denote the truncation of $\mathcal{OCP}_T(x_i)$ to the reduced horizon $T - \tilde{\tau}$. By Bellman's principle of optimality, the end pieces of $z_i^\star(\cdot, x_i)$ truncated to $[\tilde{\tau}, T]$ are optimal for $\mathcal{OCP}_{T-\tilde{\tau}}(x^\star(\tilde{\tau}, x_i, u^\star(\cdot)))$. With $x^\star(\tilde{\tau}, x_1, u^\star(\cdot, x_i)) = x^\star(\tilde{\tau}, x_2, u^\star(\cdot, x_i)) = \bar{x}$ from the choice of $\tilde{\tau}$, it follows that $\mathcal{OCP}_{T-\tilde{\tau}}(x^\star(\tilde{\tau}, x_1, u^\star(\cdot)))$ and $\mathcal{OCP}_{T-\tilde{\tau}}(x^\star(\tilde{\tau}, x_2, u^\star(\cdot)))$ are identical, which leads to (9). $\square$

[4] One may wonder whether it is possible that exact turnpike solutions first enter the turnpike exactly, then leave the turnpike, and finally return to the turnpike. However, a simple proof by contradiction shows that, in a time-invariant setting, any finite-time beneficial turnpike excursion (that returns to the turnpike) either violates the exact turnpike property or cannot exist. Due to space limitations we do not elaborate on this here.

[5] In principle, one could denote the duration of the turnpike-approaching and the turnpike-leaving part of the optimal solutions as $T_1(x_0, \bar{x})$, respectively, $T_2(\bar{x})$. In order to simplify the notation, we surpress the argument $\bar{x}$ in both cases. One should keep in mind, however, that changing the constraints or the cost function of $\mathcal{OCP}_T(x_0)$ leads, in general, to a change of the turnpike $\bar{z} = (\bar{x}, \bar{u})^T$ and thus also to different values of $T_1(x_0)$ and $T_2$.

The message of the previous lemma is as follows: If the optimal solutions to an OCP show exact turnpike behavior, then the end pieces of the optimal solutions starting at different initial conditions lead to identical cost. It follows that the duration $T_2$ of the turnpike-exiting part of the optimal solutions is independent of $x_0$ and $T$.

Next, we characterize the start pieces of exact turnpike solutions. To this end, we denote as $\mathcal{OCP}_{\hat{\tau}}(x_0, \bar{x})$ a variant of OCP (3), whereby the horizon is limited to $\hat{\tau} \in [\tau_1(x_0), \tau_2(T)]$ and the additional terminal constraint $x(\hat{\tau}) = \bar{x}$ is considered. Let $u^\star_{\hat{\tau}}(\cdot, x_0, \bar{x})$ and $z^\star_{\hat{\tau}}(\cdot, x_0, \bar{x})$ denote the optimal input and the optimal pair of $\mathcal{OCP}_{\hat{\tau}}(x_0, \bar{x})$. The following lemma shows that the first part of an exact turnpike solution has to be an optimal solution to $\mathcal{OCP}_{\hat{\tau}}(x_0, \bar{x})$.

*Lemma 2.* (Turnpikes are reached optimally).
*Let $z^\star(\cdot, x_0)$ denote an optimal pair of $\mathcal{OCP}_T(x_0)$ and let Assumption 3 hold. Then*

$$\int_0^{\hat{\tau}} F(z^\star_{\hat{\tau}}(\tau, x_0, \bar{x}))d\tau = \int_0^{\hat{\tau}} F(z^\star(\tau, x_0))d\tau \tag{10}$$

*holds for all $\hat{\tau} \in [\tau_1(x_0), \tau_2(T)]$.*

**Proof.** Note that $z^\star(\hat{\tau}, x_0) = \bar{z}$ holds for all $\hat{\tau} \in [\tau_1(x_0), \tau_2(T)]$. Hence, the truncation of $z^\star(\cdot, x_0)$ to $[0, \hat{\tau}]$ is an admissible pair of $\mathcal{OCP}_{\hat{\tau}}(x_0, \bar{x})$. Optimality of $z^\star_{\hat{\tau}}(\cdot, x_0, \bar{x})$ in $\mathcal{OCP}_{\hat{\tau}}(x_0, \bar{x})$ leads to

$$\int_0^{\hat{\tau}} F(z^\star_{\hat{\tau}}(\tau, x_0, \bar{x}))d\tau \leq \int_0^{\hat{\tau}} F(z^\star(\tau, x_0))d\tau. \tag{11a}$$

Let us consider the input

$$\hat{u}(\tau, x_0) = \begin{cases} u^\star_{\hat{\tau}}(\tau, x_0, \bar{x}) & \tau \in [0,\, \hat{\tau}) \\ u^\star(\tau, x_0) & \tau \in [\hat{\tau},\, T] \end{cases},$$

which is admissible in $\mathcal{OCP}_T(x_0)$. The corresponding admissible pair is denoted as $\hat{z}(\cdot, x_0)$. Optimality of $z^\star(\cdot, x_0)$ in $\mathcal{OCP}_T(x_0)$ leads to

$$\int_0^{T} F(z^\star(\tau, x_0))d\tau \leq \int_0^{T} F(\hat{z}(\tau, x_0))d\tau.$$

Since $z^\star(\hat{\tau}, x_0) = \hat{z}(\hat{\tau}, x_0) = \bar{z}$ and $\hat{u}(\tau, x_0) = u^\star(\tau, x_0)$ for all $\tau \in [\hat{\tau},\, T]$, the last inequality can be rewritten as

$$\int_0^{\hat{\tau}} F(z^\star(\tau, x_0))d\tau \leq \int_0^{\hat{\tau}} F(\hat{z}(\tau, x_0))d\tau. \tag{11b}$$

However, the construction of $\hat{u}(\cdot, x_0)$ implies

$$\int_0^{\hat{\tau}} F(\hat{z}(\tau, x_0))d\tau = \int_0^{\hat{\tau}} F(z^\star_{\hat{\tau}}(\tau, x_0))d\tau. \tag{11c}$$

Equality (10) follows from (11a)–(11c). □

It is worth mentioning that the solutions to $\mathcal{OCP}_{\hat{\tau}}(x_0, \bar{x})$ are, in general, not minimum-time solutions. Hence, the time $\tau_1(x_0)$ is often larger than the minimal time required to steer the state from $x_0$ to $\bar{x}$. [6]

We will show next that the exact turnpike property allows for an easy construction of optimal solutions to sequences of $\mathcal{OCP}_T(x_p(t_k))$ as they arise in the context of NMPC.

Consider

$$x_\delta := x^\star(\delta, x_0, u^\star(\cdot, x_0)),$$

which is the state reached from $x_0$ after one sampling time upon application of the optimal input $u^\star(\cdot, x_0)$. We want to show that the exact turnpike property allows constructing the optimal solution to $\mathcal{OCP}_T(x_\delta)$ from the optimal solution to $\mathcal{OCP}_T(x_0)$. To this end, we assume that $T \geq T_{min}$ and consider the following input trajectory

$$u(\tau, \delta) = \begin{cases} u^\star(\tau + \delta, x_0) & \tau \in [0,\, \tau_2(T) - \delta) \\ u^\star(\tau, x_0) & \tau \in [\tau_2(T) - \delta,\, T] \end{cases}. \tag{12}$$

*Theorem 1.* (Optimal recursive solution to $\mathcal{OCP}_T(x_0)$).
*Let Assumption 3 hold. Then, for any $\delta$ satisfying*

$$0 \leq \delta < \min\{\tau_1(x_0),\, \tau_2(T) - \tau_1(x_0)\}, \tag{13}$$

*the input trajectory $u(\cdot, \delta)$ from* (12) *is an optimal solution to $\mathcal{OCP}_T(x_\delta)$.*

**Proof.** Consider $\mathcal{OCP}_{T+\delta}(x_0)$, whereby, in comparison to $\mathcal{OCP}_T(x_0)$, the horizon is increased from $T$ to $T + \delta$. Two observations are important: (a) We know from Lemma 2 that, for all $\tau \in [0,\, \tau_2(T)]$, the input $u^\star_{\tau_2(T)}(\tau, x_0, \bar{x})$ from $\mathcal{OCP}_{\tau_2(T)}(x_0, \bar{x})$ can be used as the first part of the optimal input in $\mathcal{OCP}_{T+\delta}(x_0)$; and (b) we can infer from Lemma 1 that for all $\tau \in [\tau_1(x_0) + \delta,\, T + \delta]$ the input $u^\star(\tau - \delta, x_0)$ stemming from $\mathcal{OCP}_T(x_0)$ can be used as the second part of the optimal input in $\mathcal{OCP}_{T+\delta}(x_0)$. Combining these observations gives the optimal input in $\mathcal{OCP}_{T+\delta}(x_0)$

$$u^\star_{T+\delta}(\tau, x_0) = \begin{cases} u^\star_{\tau_2(T)}(\tau, x_0, \bar{x}) & \tau \in [0,\, \bar{\tau}) \\ u^\star(\tau - \delta, x_0) & \tau \in [\bar{\tau},\, T + \delta] \end{cases}, \tag{14}$$

with $\bar{\tau} \in [\tau_1(x_0) + \delta, \tau_2(T)]$. Lemma 2 indicates that, for all $\tau \in [0, \tau_2(T)]$, the optimal input in $\mathcal{OCP}_{\tau_2(T)}(x_0, \bar{x})$ can be replaced, without loss of optimality, by $u^\star(\tau, x_0)$ from $\mathcal{OCP}_T(x_0)$. Thus, setting

$$u^\star_{\tau_2(T)}(\tau, x_0, \bar{x}) = u^\star(\tau, x_0)$$

for all $\tau \in [0,\, \bar{\tau})$ in (14) gives

$$u^\star_{T+\delta}(\tau, x_0) = \begin{cases} u^\star(\tau, x_0) & \tau \in [0,\, \bar{\tau}) \\ u^\star(\tau - \delta, x_0) & \tau \in [\bar{\tau},\, T + \delta]. \end{cases} \tag{15}$$

Now observe that $x^\star(\delta, x_0, u^\star_{T+\delta}(\cdot, x_0)) = x_\delta$. Hence, by Bellman's principle of optimality, for all $\tau \in [0, T]$, the input $u^\star_{T+\delta}(\tau + \delta, x_0)$ is optimal for $\mathcal{OCP}_T(x_\delta)$. Rewriting (15) in terms of $\mathcal{OCP}_T(x_\delta)$ gives

$$u^\star(\tau, x_\delta) = \begin{cases} u^\star(\tau + \delta, x_0) & \tau \in [0,\, \bar{\tau} - \delta) \\ u^\star(\tau, x_0) & \tau \in [\bar{\tau} - \delta,\, T] \end{cases}. \tag{16}$$

Note that, for $\bar{\tau} = \tau_2(T)$, the last equation corresponds to (12), which indicates that $u(\cdot, \delta)$ from (12) is *optimal* for $\mathcal{OCP}_T(x_\delta)$. □

*Corollary 1. Consider $\mathcal{OCP}_T(x_i)$ with $x_i \in \{x_0, x_\delta\}$, with $\delta$ from* (13)*, and let Assumption 3 hold. Then, $\tau_1(x_\delta) = \tau_1(x_0) - \delta$.*

**Proof.** The proof follows from Theorem 1 and the following observation: The construction of $u(\cdot, \delta)$ in (12) implies that, at $\tau = \tau_1(x_0) - \delta$, the optimal solution satisfies $x^\star(\tau, x_\delta, u(\cdot, \delta)) = \bar{x}$, whereas, for $0 \leq \tau < \tau_1(x_0) - \delta$, $x^\star(\tau, x_\delta, u(\cdot, \delta)) \neq \bar{x}$. □

*Corollary 2. Consider $\mathcal{OCP}_T(\bar{x})$, where $\bar{x}$ is the turnpike steady state, and let Assumption 3 hold. Then, $x^\star(\delta, \bar{x}, u^\star(\cdot, x_\delta)) = \bar{x}$.*

The proof follows directly from Theorem 1.

[6] An easy example demonstrating this is obtained by comparing the minimum optimal control for the double integrator to the optimal solution to Fuller's problem, see Liberzon (2012).

## 4. CONVERGENCE OF NMPC BASED ON TURNPIKE PROPERTIES

Theorem 1 and Corollaries 1 and 2 indicate that (non-pathological) exact turnpike properties allow easy construction of optimal solutions to receding-horizon sequences of $\mathcal{OCP}_T(x_p(t_k))$ as they arise in NMPC. Hence, we use exact turnpike properties to establish finite-time convergence and recursive feasibility for sampled-data NMPC. Consider the plant (1) controlled by the NMPC scheme based on $\mathcal{OCP}_T(x_p(t_k))$ that generates the input (4).

*Theorem 2.* (Convergence of NMPC via exact turnpike). *Let Assumptions 1–2 hold and $\mathcal{OCP}_T(x_0)$ satisfy Assumption 3. Then, there exists a sampling time $\delta \in (0, \tau_2(T) - \tau_1(x_0))$ such that the following properties hold:*

*(i) If $\mathcal{OCP}_T(x_0)$ is initially feasible, then it is feasible for all subsequent sampling instants $t_k > 0$.*
*(ii) There exists a finite time $\bar{t} \geq 0$ such that, for any $x_0 \in \mathcal{X}_0$, the NMPC input (4) generates*

$$x_p(t, x_0, u_p(\cdot)) = \bar{x}, \quad \forall t \geq \bar{t}.$$

**Proof.** The proof proceeds in three steps: Step 1 establishes the existence of $\delta \in (0, \tau_2(T) - \tau_1(x_0))$, Step 2 shows recursive feasibility, while Step 3 establishes finite-time convergence.

*Step 1 (Existence of a sampling time $\delta$):* Theorem 1 shows that, given a fixed $x_0 \in \mathcal{X}_0$, the optimal input $u^\star(\cdot, x_0)$ and a sampling time $\delta$ satisfying (13), the input $u(\cdot, \delta)$ in (12) is an optimal solution to $\mathcal{OCP}_T(x_\delta)$. Hence, for each $x_0 \in \mathcal{X}_0$, the choice of a suitable $\delta$ is dictated by (13). Consider

$$\tau_1^{max} := \max_{x_0 \in \mathcal{X}_0} \tau_1(x_0).$$

Since $\mathcal{X}_0$ is a compact set and, for all $x_0$, Assumption 3 implies $\tau_1(x_0) < T < \infty$, we have that $\tau_1^{max} < \infty$. Assumption 3 also gives $\tau_2(T) - \tau_1^{max} > 0$. Hence, for any sampling time $\delta$ satisfying $0 < \delta < \min\{\tau_1^{max}, \tau_2(T) - \tau_1^{max}\}$, Theorem 1 holds for all $x_0 \in \mathcal{X}_0$ using the same $\delta$.

*Step 2 (Recursive feasibility):* Assume that $\mathcal{OCP}_T(x_p(t_k))$ is feasible and $u^\star(\cdot, x_p(t_k))$ is applied for $t \in [t_k, t_k + \delta)$. Due to Assumption 1 (no plant-model mismatch), the plant state and the predicted state at time $t_{k+1} = t_k + \delta$ are equal, i.e., $x_p(t_{k+1}) = x^\star(\delta, x_p(t_k), u^\star(\cdot, x_p(t_k)))$. It follows from Theorem 1 that $u(\cdot, x_p(t_k))$ in (12) is an optimal solution to $\mathcal{OCP}_T(x_p(t_{k+1}))$. Hence, the sequence of $\mathcal{OCP}_T(x_p(t_k))$ problems is recursively feasible for all $t_k$.

*Step 3 (Finite-time convergence):* By Assumption 1, it follows from Corollary 1 that

$$\tau_1(x_p(t_{k+1})) = \tau_1(x_p(t_k)) - \delta, \quad \forall k \in \{0, 1, \dots, \bar{k} - 1\}$$

where $\bar{k}$ is given by

$$\bar{k} = \underset{i}{\operatorname{argmin}} \{i \in \mathbb{N} : \tau_1(x_0) - i\delta \leq 0\}.$$

The last two equations show that, for $k := \bar{k}$, we have

$$x_p(t_{\bar{k}}) = \bar{x}.$$

So far, we have shown that the NMPC scheme based on OCP (3) and initialized at $x_0$ reaches the turnpike steady state $\bar{x}$ at time $\bar{t} = \tau_1(x_0)$. Furthermore, we can infer from Corollary 2 that the generated trajectory of (1) satisfies $x_p(t, \bar{x}, u_p(\cdot)) = \bar{x}$ for all $t \geq \tau_1(x_0)$. This completes the proof. □

Theorem 2 shows that exact turnpike properties allow establishing recursive feasibility and finite-time convergence of the sampled-data NMPC scheme (3)–(4). Note that, provided a reachability assumption holds, the most attractive steady-state value $\bar{x}$ corresponds to an *optimal* solution to (8). Note also that no assumption is made on the cost function $F$ being lower bounded by a distance measure to a setpoint. Again, we want to re-emphasize that the conditions of Theorem 2 do not require any terminal penalty nor a terminal region constraint.

*Remark 5.* (Verification of turnpikes properties).
At this point it is fair to ask for conditions that allow verifaction of turnpike properties. In Faulwasser et al. (2014) it shown that a dissipativity condition combined with a reachability assumption are sufficient to guarantee the existence of (state) turnpike properties. [7] It is also shown that, for polynomial OCPs, sum-of-squares programming can be used to verify the required dissipativity inequality. Given the existence of a turnpike, verification of the exactness property requires a detailed analysis of the underlying OCP. First steps in this direction have shown that, for OCPs that exhibit singular arcs at steady state, turnpikes, if they exist, are exact, cf. Faulwasser and Bonvin (2015). Due to space limitations we do not detail this here.

## 5. EXAMPLE – OPTIMAL FISH HARVEST

To illustrate our previous findings we consider an example simple enough for direct calculation of optimal solutions. The task at hand is to minimize the objective

$$J_T(x_0, u(\cdot)) = \frac{1}{T} \int_0^T \big[ax(\tau) + bu(\tau) - cx(\tau)u(\tau)\big]\, d\tau, \tag{17a}$$

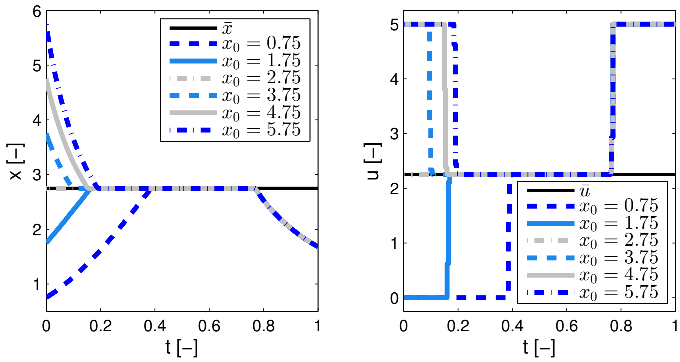
where $x, u \in \mathbb{R}$, and the dynamics of $x$ are given by

$$\dot{x} = x(x - x_s - u), \quad x(0) = x_0 > 0. \tag{17b}$$

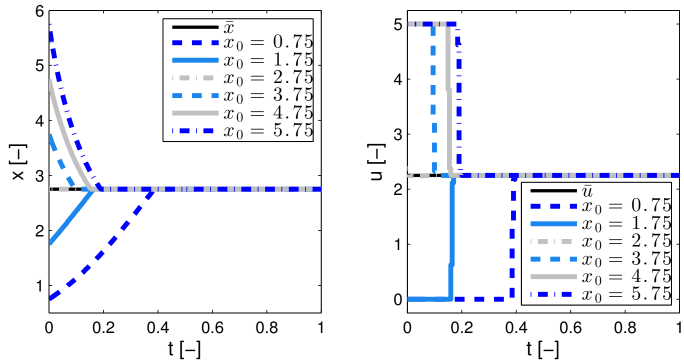
The state $x$ is the fish density in a certain habitat, the control $u$ is the fishing rate, and the parameter $x_s$ describes the highest sustainable fish density. In slightly modified form, i.e., with an additional terminal constraint, this OCP is analyzed in Cliff and Vincent (1973). We consider the task of minimizing (17a) subject to (17b) and the constraints for all $\tau \in [0, t| : u(\tau) \in [0, u_{max}]$ and $x(\tau) \geq x_{min}$, with $x_s > x_{min} > 0$. The parameter values are $a = 1, b = c = 2, x_s = 5, u_{max} = 5, x_{min} = 0.1$. It can be shown that this singular OCP has exactly one singular arc along which $x_{sing} = \frac{1}{2}x_s + \frac{b-a}{2c}, u_{sing} = x_s - x_{sing}$. In other words, along the singular arc, the optimal pair $z^\star(\cdot, x_0)$ is at steady state.

Figure 2a shows the optimal solutions of the fish harvest problem for different intial conditions $x_{0,i} = 0.75 + i$ and a fixed horizon $T = 1$. As one can see, the optimal solutions show exact turnpike behavior. It turns out that the turnpike is the singular arc, i.e., $\bar{z} = (x_{sing}, u_{sing})^T$. One can also observe that, as predicted by Lemma 1, the end pieces of the turnpike solutions are all identical. Figure 2b depicts the closed-loop solutions obtained by solving

[7] A slighty different property is shown therein, instead of conditions ensuring input-state turnpikes, conditions for state turnpikes are discussed. Note that with straightforward modifications the proof presented in Faulwasser et al. (2014) holds for input-state turnpikes.




(a) Optimal solutions to OCP (17) for different $x_0$.




(b) Closed-loop NMPC solutions for different $x_0$.

Fig. 2. Simulation results for the fish harvest problem.

the OCP in a receding horizon fashion. The prediction horizon is $T = 1$. As one can see, and as predicted by Theorem 2, the closed-loop NMPC solutions reach the turnpike in finite time $T_1(x_0)$ and remain there.

## 6. CONCLUSION

This paper has presented novel sufficient convergence conditions for sampled-data (economic) NMPC schemes with input and state constraints. The proposed conditions, which are based on an exact turnpike property, do not require terminal penalties nor terminal constraints and, furthermore, they ensure recursive feasibility. Future work will investigate conditions guaranteeing the exactness of turnpike properties and the generalization of the convergence conditions to approximate turnpikes.

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