Analysis of Exclusively-kinetic Two-link Underactuated Mechanical Systems

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Abstract

Analysis of exclusively-kinetic two-link underactuated mechanical systems is undertaken in this paper. It is first shown that such systems are not full-state feedback linearizable around any equilibirium point. Also, the equilibrium points for which the system is small-time locally controllable (STLC) is at most a one dimensional submanifold. A concept less restrictive than STLC, termed the small-time local output controllability (STLOC) is introduced, the satisfaction of which guarantees that a chosen configuration output can be controlled at its desired value. It is shown that the class of systems considered are STLOC, if the inertial coupling between the input and output is non-zero. Also, in such a case, the system is nonminimum phase (NMP). An example section illustrates all results presented.

Keywords: Underactuated mechanical system, small-time local controllability, nonminimum-phase, configuration output

1 Introduction

Underactuated mechanical systems have recently gained research attention due to the variety of new problems they have generated [3,10,12]. In most cases, simple control methodologies fail and sophisticated control techniques based on heuristics and empirical observation have been used for the control of such systems [6,7]. Analysis of a class of such underactuated systems is undertaken in this paper. For the sake of simplicity, the analysis is restricted to two-link underactuated systems under the assumption that gravity and friction are absent. This class has received quite some attention, see [2,12] and references therein.

The following notions play a key role in this paper:

- *Full-state feedback linearizability (FL):* This property implies that the nonlinear system can be transformed into a linear one with a nonlinear feedback and a state transformation [5]. Then, standard linear control techniques can be used for controller design.
  It is shown here that the two-link exclusively kinetic underactuated systems do not satisfy the necessary and sufficient conditions for full-state feedback linearizability.
- *Small-time local controllability (STLC):* This is an extension to the well understood concept of Kalman controllability to nonlinear systems. It implies that a state arbitrarily close to the initial one can be reached in arbitrarily small time. It is a property for which there are some tractable conditions to assess it [13].
  Though it has been shown in [2] that a planar 2R robot does not satisfy the sufficient conditions for STLC, the work is inconclusive towards establishing or refuting this property. Herein, exclusively-kinetic two-link underactuated mechanical systems are shown not to be STLC.
- *Small-time local output controllability (STLOC):* STLC is a fairly restrictive concept and a system need not necessarily be STLC to keep a chosen configuration output at its desired value. In this study, a relatively new concept of small-time local output controllability (STLOC) is introduced, which only demands that a neighbourhood of the equilibirium in the output space can be reached by manipulating the inputs. Only configuration outputs will be considered, i.e. outputs that do not depend on time derivatives of the generalized coordinates. The good news is that except for some pathological cases, exclusively-kinetic two-link underactuated mechanical systems can be shown to be STLOC almost everywhere.
- *Nonminimum phase (NMP):* The bad news, however, is that in most cases the system is nonminimum phase. A formal link between two-link exclusively-kinetic underactuated systems and nonminimum-phase systems has been initiated in [8]. The nonminimum-phase property is an input-output property which is associated with right-half-plane transmission zeros in the case of linear systems. However, in the nonlinear scenario, nonminimum-phase systems are defined based on the stability of the zero dynamics [1].

The paper is organized as follows. Section 2 introduces certain preliminaries and new definitions. Rigid body dynamics formulation for two-link systems is also given. The analysis of full-state feedback linearizability is done in Section 3, that of STLC in Section 4, and STLOC in Section 5. Section 6 studies nonminimum-phase characteristics that appear in exclusively-kinetic two-link underactuated mechanical systems. Finally, Section 7 presents examples to illustrate the above concepts.

2 Preliminaries

Consider the affine-in-input system,

$$\dot{x} = f(x) \; + \; g(x)\, u, \quad y = h(x), \quad x(0) = x_0 \tag{1}$$

where $x \in \mathcal{X} \subset R^n$ and $f(0) = 0$ (i.e., $x = 0$ is an equilibrium point). In this paper, the SISO setting will be considered, i.e., $u \in \mathcal{U} \subset R$ and $y \in Y \subset R$. Moreover f and g and all other vector fields will be considered analytic unless otherwise specified. For a given input $u : [0, t] \to \mathcal{U}$, let the solution to (1) starting from the initial condition x_0 and evaluated at time t be denoted by $x(t, x_0, u)$.

2.1 Full-state Feedback Linearizability (FL)

The concepts pertaining to feedback linearization are defined in the literature and the reader is referred to standard texts on nonlinear control for these definitions (see for example [5]). Nevertheless, for easy reference, the necessary and sufficient conditions of feedback linearization will be recalled.

Theorem 1 *System (1) is state feedback linearizable around a point x_0 iff the following two conditions are satisfied: (i) the matrix $[g(x_0),\ fg(x_0),\ \cdots,\ f^{n-1}g(x_0)]$* [1] *has rank n and (ii) the distribution* span$\{g,\ fg,\ \cdots,\ f^{n-2}g\}$ *is involutive* [2] *in a neighborhood of x_0.*

2.2 Small-time Local Controllability (STLC)

Though a sufficient condition for STLC is presented in [13], the following necessary condition will be used in this paper:

Theorem 2 *If system (1) is STLC at x_0, then $gfg(x_0) \in \mathcal{S}^1(x_0) = span\{g(x_0), fg(x_0), ffg(x_0), \ldots\}$.*

$\mathcal{S}^1$ is the space spanned by tangent vectors stemming from brackets containing at most one g and arbitrarily many f.

[1] In this paper, the following notation will be used for the Lie brackets: $[v_1, v_2] = v_1 v_2$ and $[v_1, [v_2, v_3]] = v_1 v_2 v_3$.

[2] Closed under Lie braketing

2.3 Small-time Local Output Controllability (STLOC)

An new controllability definition is introduced (see also [4]). It restricts the concept of local controllability to the output space.

Definition 3 *System (1) is small-time localy output controllable from x_0 if $h(x_0)$ is in the interior of the reachable set $\mathcal{R}_T^Y(x_0) = \{y \in \mathcal{Y} \mid \exists\ u : [0, t] \to \mathcal{U}, t \in [0, T]$, such that $y = h(x(t, x_0, u)),\ \ h(x(\tau, x_0, u)) \in Y, 0 \le \tau \le t\}$ for all neighbourhoods Y and for all $T > 0$.*

2.4 Nonminimum-phase (NMP) Systems

Nonminimum phase systems are defined based on the *non* asymptotic stability of the internal dynamics [1].

Definition 4 *System (1) is (nonminimum) minimum phase at x^*, if x^* is (not) an asymptotically stable equilibrium point of the internal dynamics*

$$\dot{x} \; = \; f^*(x) \; = \; f(x) \; + \; g(x)\, u^*(x), \qquad x \in H^* \tag{2}$$

where H^* is the manifold where $h(x) = h^*$, and $u^* : H^* \to \mathbf{R}$ is such that $f(x) + g(x)\, u^*(x)$ is tangent to H^*.

It could be more appropriate to use the term "constant output induced dynamics" in the present context, since it is in-between the general definition of internal dynamics ($h(x) = h_{ref}(t)$) and the traditional definition of zero dynamics ($h(x) = 0$). However, the term "internal dynamics" will be retained in the sequel, since it makes the explanations clearer.

2.5 Two-link Underactuated Mechanical Systems

Definition 5 *A system is underactuated if it has fewer independent actuators than its degree of freedom [11].*

Definition 6 *A two-link mechanical system is a set of two rigid bodies connected to each other by an articulation, and one of the bodies is connected to a base frame through another articulation. One degree of freedom is allowed for each articulation. A two-link underactuated mechanical system is a two-link system which is underactuated.*

A natural choice of generalized coordinates correspond to the link coordinates themselves. It specifies the position of the next link with respect to the previous one along the movement permitted by the joint associated to it. The rigid body dynamics of a two-link exclusively-kinetic mechanical system in the link coordinates are given by [7]:

$$\begin{aligned}
\ddot{q}_1 &= -\Gamma^1_{11}\dot{q}_1^2 - 2\Gamma^1_{12}\dot{q}_1\dot{q}_2 - \Gamma^1_{22}\dot{q}_2^2 + n_{11}\tau_1 + n_{12}\tau_2 \\
\ddot{q}_2 &= -\Gamma^2_{11}\dot{q}_1^2 - 2\Gamma^2_{12}\dot{q}_1\dot{q}_2 - \Gamma^2_{22}\dot{q}_2^2 + n_{21}\tau_1 + n_{22}\tau_2
\end{aligned} \tag{3}$$

where $q = [q_1, q_2]^T$, n_{mk} denote the elements of the inverse of the inertia matrix $N = D^{-1}$, and Γ_{ij}^m, $m = 1, 2$ are the Christoffel symbols of the second kind [7].

In the sequel, results will be obtained for any kind of output not depending on the velocities $\dot{q}_1$ and $\dot{q}_2$.

Definition 7 *A configuration output, $c(q_1, q_2)$, is a submersion of the configuration space $\{q_1, q_2\}$ into a one-dimensional manifold.*

For a given configuration output $c(q_1, q_2)$, let $\xi(q_1, q_2)$ be another submersion such that ξ and c form a set of generalized coordinates. This means that the transformation $\mathcal{T} : [q_1\ q_2]^T \leftrightarrow [c\ \xi]^T$ is a diffeomorphism with a non-singular Jacobian. The dynamic equations (3) can be written in the new coordinates as:

$$\begin{aligned} \ddot{c} &= -\Upsilon_c \dot{c}^2 - 2\Upsilon_{c\xi}\dot{c}\dot{\xi} - \Upsilon_\xi \dot{\xi}^2 + \sigma_c \tau \\ \ddot{\xi} &= -\Psi_c \dot{c}^2 - 2\Psi_{c\xi}\dot{c}\dot{\xi} - \Psi_\xi \dot{\xi}^2 + \sigma_\xi \tau \end{aligned} \tag{4}$$

where Υ_c, $\Upsilon_{c\xi}$, Υ_ξ, σ_c, Ψ_c, $\Psi_{c\xi}$, Ψ_ξ, and σ_ξ are appropriately defined functions of c and ξ. $\tau = [\tau_1\ \tau_2]\ e$, where e determines which joint is actuated. For example, if $\tau = \tau_1$, $e = [1\ 0]^T$. Note that the choice of e influences only the functions σ_c and σ_ξ.

3 Analysis of Feedback Linearizability

In this section, the structure of the Lie algebra at the equilibrium point will be analyzed first. It will then be shown that two-link exclusively-kinetic underactuated mechanical systems are not feedback linearizable around any equilibirium point.

Consider the state vector $x = [c, \xi, \dot{c}, \dot{\xi}]^T$ and the input $u = \tau$. Then, from (4), the vector fields f and g read:

$$f = \begin{bmatrix} \dot{c} \\ \dot{\xi} \\ -\Upsilon_c \dot{c}^2 - 2\Upsilon_{c\xi}\dot{c}\dot{\xi} - \Upsilon_\xi \dot{\xi}^2 \\ -\Psi_c \dot{c}^2 - 2\Psi_{c\xi}\dot{c}\dot{\xi} - \Psi_\xi \dot{\xi}^2 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ \sigma_c \\ \sigma_\xi \end{bmatrix} \tag{5}$$

The equilibrium states correspond to any choice with $\dot{c} = \dot{\xi} = 0$, i.e., define the two-dimensional equilibrium manifold $\mathcal{X}^* = \{x^* \mid x^* = [c^*, \xi^*, 0, 0]^T, c^* \in \mathbf{R}, \xi^* \in \mathbf{R}\}$. The following lemma gives an idea of the Lie brackets giving non vanishing vectors after evaluation at the equilibrium.

Lemma 8 *Let $B(k,l)$ denote a bracket with k and l being the number of times the vector fields g and f appear in the bracket, respectively. Then, $B(k,l)$ has the following structure:*

$$B(k,l) = \begin{bmatrix} \mathcal{O}_{2\times1}(l-k) \\ \mathcal{O}_{2\times1}(l-k+1) \end{bmatrix} \tag{6}$$

where $\mathcal{O}_{2\times1}(p)$ represents a matrix of dimension 2×1, where each element is a p^{th}-order function of $\dot{c}$ and $\dot{\xi}$ (e.g., $\mathcal{O}(-1) = 0$, $\mathcal{O}(0)$: function independent of $\dot{c}$ and $\dot{\xi}$, $\mathcal{O}(1) = \alpha\dot{c} + \beta\dot{\xi}$, $\mathcal{O}(2) = \alpha\dot{c}^2 + \beta\dot{c}\dot{\xi} + \gamma\dot{\xi}^2$, etc.).

PROOF. Proof by induction. By inspection, it can be seen that f and g satisfy (6). Suppose $B(k,l)$ is a bracket which satisfies (6). Then, the proof is complete if we show that Lie-bracketing with f augments the orders by 1 and that with g reduces the orders by 1.

$$[f, B] = \begin{bmatrix} \mathcal{O}_{2\times2}(l-k) & \mathcal{O}_{2\times2}(l-k-1) \\ \mathcal{O}_{2\times2}(l-k+1) & \mathcal{O}_{2\times2}(l-k) \end{bmatrix} \begin{bmatrix} \mathcal{O}_{2\times1}(1) \\ \mathcal{O}_{2\times1}(2) \end{bmatrix} - \begin{bmatrix} 0_{2\times2} & I_{2\times2} \\ \mathcal{O}_{2\times2}(2) & \mathcal{O}_{2\times2}(1) \end{bmatrix} \begin{bmatrix} \mathcal{O}_{2\times1}(l-k) \\ \mathcal{O}_{2\times1}(l-k+1) \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{2\times1}(l-k+1) \\ \mathcal{O}_{2\times1}(l-k+2) \end{bmatrix} \tag{7}$$

$$[g, B] = \begin{bmatrix} \mathcal{O}_{2\times2}(l-k) & \mathcal{O}_{2\times2}(l-k-1) \\ \mathcal{O}_{2\times2}(l-k+1) & \mathcal{O}_{2\times2}(l-k) \end{bmatrix} \begin{bmatrix} 0_{2\times1} \\ \mathcal{O}_{2\times1}(0) \end{bmatrix} - \begin{bmatrix} 0_{2\times2} & 0_{2\times2} \\ \mathcal{O}_{2\times2}(0) & \mathcal{O}_{2\times2}(0) \end{bmatrix} \begin{bmatrix} \mathcal{O}_{2\times1}(l-k) \\ \mathcal{O}_{2\times1}(l-k+1) \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{2\times1}(l-k-1) \\ \mathcal{O}_{2\times1}(l-k) \end{bmatrix} \tag{8}$$

Since $\mathcal{O}(p) = 0$ at the equilibrium for all $p > 0$, only elements of type $\mathcal{O}(0)$ need to be considered. The tangent vectors obtained by evaluating the brackets at an equilibrium point span the vector subspace associated to the third and fourth rows, if the associated brackets satisfy $k = l + 1$. The brackets to be considered are $g, gfg, gfgfg, gfgfgfg, \cdots$. Similarly, to span the vector subspace associated to the first and the second rows, one needs $k = l$, with the necessary Lie brackets being $fg, fgfg, fgfgfg, \cdots,$. More importantly, span$\{g(x^*), fg(x^*)\ , \cdots, f^{n-1}g(x^*)\}$ obtained at any equilibrium point x^* has dimension 2.

Theorem 9 *Two-link exclusively-kinetic underactuated mechanical systems are not full-state feedback linearizable at all equilibrium points.*

PROOF. The proof follows since the first condition of feedback linearizablity is not satisfied, rank$[g(x^*), fg(x^*), \cdots, f^{n-1}g(x^*)] = 2 \neq n = 4$.

4 Analysis of Small-time Local Controllability

Using the structure of the Lie brackets discussed in the previous section, it will be shown here that two-link exclusively-kinetic underactuated mechanical systems are not small-time locally controllable outside a set of measure zero. Particularly, the vector field gfg plays a very crucial role as will be discussed in the following lemma.

Lemma 10 *If $gfg \propto g$ [3], then $(gfgf \cdots g) \propto g$ and $(fgf \cdots g) \propto fg$.*

[3] Given two vector fields v_1 and v_2, $v_1 \propto v_2$ means that there exists $\alpha : \mathbf{R}^n \to \mathbf{R}$, such that $v_1(x) = \alpha(x) v_2(x)$.

PROOF. The proof is by induction. The hypothesis of the lemma states $gfg \propto g$. Suppose $(gfgf\cdots g) \propto g$. Then $[f,(gfgf\cdots g)] \propto [f,g]$. Also $[g,[f,(gfgf\cdots g)]] \propto [g,[f,g]] \propto g$.

Theorem 11 *The submanifold of equilibria points for which the system is STLC is at most one dimensional.*

PROOF. Due to the structure of the Lie brackets explained in Lemma 8, the vector subspace spanned by the tangent vectors obtained after evaluating at the equilibrium the brackets having at most one g, $\mathcal{S}^1(x^*)$, is generated by only two vectors: $\mathcal{S}^1(x^*) = \text{span}\{g(x^*), fg(x^*)\}$ with

$$g(x^*) = [0, 0, \sigma_c(c^*, \xi^*), \sigma_\xi(c^*, \xi^*)]^T \quad (9)$$

$$fg(x^*) = [\sigma_c(c^*, \xi^*), \sigma_\xi(c^*, \xi^*), 0, 0]^T. \quad (10)$$

Whether or not the system is STLC at x^* depends on the inclusion of $gfg(x^*)$ in the vector subspace $\mathcal{S}^1(x^*)$. Consider the function $R(x) = \text{rank}[g(x), fg(x), gfg(x)]$. Choose a 2 dimensional submanifold $\mathcal{W}$ of the equilibrium manifold $\mathcal{X}^*$ containing x^*, such that the set $\mathcal{V} = \{x^* \mid R(x^*) = 0\} \cap \mathcal{W}$ defines a submanifold. Notice that $gfg(x^*) \in \mathcal{S}^1(x^*)$ for all points $x^* \in \mathcal{V}$. Two cases need to be envisaged:

(1) $\dim \mathcal{V} = 2$.
Since R is an analytic function, $R = 0$ everywhere in the manifold $\mathcal{W}$ and also in $\mathcal{V}$, i.e. $gfg(x^*) = \alpha g(x^*)$. This in turn implies $gfg \propto g$ since both gfg and g do not depend on velocities. Lemma 10 then shows that all tangent vectors that can potentially augment the rank at the equilibrium x^* are in $\mathcal{S}^1(x^*)$. Since $\dim\, \mathcal{S}^1(x^*) = 2 < 4$ the system is not accessible at x^* ([5]). Since accessibility is a prerequisite for STLC (see [13]), the system is not STLC at all equilibrium points.

(2) $\dim \mathcal{V} \le 1$.
The system cannot be STLC when $x^* \notin \mathcal{V}$, since in such a case there exists points $x^* \in \mathcal{W}$ such that $x^* \notin \mathcal{V}$ and $gfg(x^*) \notin \mathcal{S}^1(x^*)$. After applying the contraposition of Theorem 2, the system is not STLC at these points.

Remark 12 *Notice that when dim $\mathcal{V} = 1$, the equilibria submanifold $\mathcal{V}$ for which the system might be STLC is not dense in the equilibrium manifold. This statement is not proven here.*

5 Analysis of Small-time Local Output Controllability

Though in the earlier section, it is shown that exclusively-kinetic two-link underactuated systems lose the STLC property almost everywhere, the situation is not as bad as one would imagine, since the output is controllable for most configurations. This result is formulated in the next theorem.

Theorem 13 *The system (4) is STLOC at an equilibrium if $\sigma_c(c^*, \xi^*) \neq 0$.*

PROOF. The functional expansion described in [9] is applied here to give the evolution of the system's output $c(t)$ under the input $\tau(t)$. The expansion is done at the equilibrium point $x^* = [c^*, \xi^*, 0, 0]^T$. Since small-time controllability deals with the time instants $0 < t < \epsilon$ as $\epsilon \to 0$, it is sufficient to consider a constant input, i.e $\tau(t) = \bar{\tau}$ with $\bar{\tau} \in \mathbf{R}$. For the sake of this proof, the notation $\dot{x} = f(x) + g(x)\tau = \rho_0(x) + \rho_1(x)\tau$ will be used, since it keeps the functional notation compact. The functional expansion reads

$$c(t) = c^* + \sum_{k=0}^{\infty} \sum_{i_0,\ldots,i_k=0}^{1} (L_{\rho_{i_0}} \ldots L_{\rho_{i_k}} c)(x^*) \frac{t^k}{k!} \bar{\tau}^{\sum_0^k i_k} \quad (11)$$

It can be worked out that the Lie derivatives $L_f c = L_g c = L_f L_f c = L_f L_g c = L_g L_g c = 0$ and $L_g L_f c = \sigma_c$. So, the output evolution under constant input reads:

$$c(t) = c^* + \frac{\sigma_c}{2} \bar{\tau} t^2 + O(t^3). \quad (12)$$

If $\sigma_c \neq 0$, the second term in (12) dominates the other ones, for a sufficiently small t. Then, by changing the sign of $\bar{\tau}$, both $c(t) > c^*$ and $c(t) < c^*$ can be reached in small time. So, the system is STLOC.

From Theorem 13, it can be seen that a necessary condition for the loss of STLOC is the absence of direct transmitted torque from the actuator to the output of interest ($\sigma_c = 0$), either due to the geometry of the axis or due to the "bad" configuration. In intuitive terms, only quadratic effects, i.e., centrifugal and Coriolis, are present and they are most of the times insufficient to steer the system in a whole neighbourhood around the equilibrium.

Theorem 14 *If $\sigma_c(c^*, \xi^*) = 0$ and $\frac{\partial \sigma_c}{\partial \xi} \neq 2\sigma_\xi \Upsilon_\xi$, system (4) is not STLOC.*

PROOF. As in the case of the previous theorem the functional expansion (11) is used. With $\sigma_c(c^*, \xi^*) = 0$ the derivative $L_g L_f c = \sigma_c$ goes to zero. Also, it can be worked out that the Lie derivatives of order three, i.e., $L_{\rho_{i_0}} L_{\rho_{i_1}} L_{\rho_{i_2}} c$ for $i_j = 0, 1$, vanish at x^*. Among the 16 Lie derivatives of order four, $L_{\rho_{i_0}} L_{\rho_{i_1}} L_{\rho_{i_2}} L_{\rho_{i_3}} c$, the only two derivatives that do not vanish at x^* are $L_g L_g L_f L_f c = -2\sigma_\xi^2 \Upsilon_\xi$ and $L_g L_f L_g L_f c = \sigma_\xi \frac{\partial \sigma_c}{\partial \xi}$. So, the output evolution under constant input reads:

$$c(t) = c^* + \frac{\sigma_\xi}{24} \left(\frac{\partial \sigma_c}{\partial \xi} - 2\sigma_\xi \Upsilon_\xi \right) \bar{\tau}^2 t^4 + O(t^5). \quad (13)$$

Note that due to the positive definiteness of the inertia matrix $\sigma_\xi \neq 0$, when $\sigma_c = 0$. When $\frac{\partial \sigma_c}{\partial \xi} \neq 2\sigma_\xi \Upsilon_\xi$, the second term in (13) dominates the other ones, for sufficiently small t. Then, depending on the sign of $\left(\frac{\partial \sigma_c}{\partial \xi} - 2\sigma_\xi \Upsilon_\xi\right)$, either $c(t) \geq c^*$ or $c(t) \leq c^*$ can be reached but never both. Thus, c^* is not in the interior of the reachable set for small enough t.

Remark 15 *To illustrate the gap between the necessary and sufficient conditions of STLOC, two cases with* $\sigma_c = 0$ *and* $\frac{\partial \sigma}{\partial \xi} = 2\sigma_\xi \Upsilon_\xi$ *will be given in the example section. One of them will be shown to be STLOC while the other is not.*

6 Analysis of Minimum-phase Behavior

The analysis of minimum-phase behavior requires the study of the dynamics induced by a constant output. The structure of the internal dynamics will be given in Lemma 16. The main result on minimum phase is based on the instability of such generic dynamics.

Lemma 16 *Consider the two-link exclusively-kinetic underactuated nonlinear system (4) with configuration output, c. If* $\sigma_c(c^*, \xi^*) \neq 0$*, the internal dynamics has the structure*

$$\ddot{\xi} = K(\xi)\dot{\xi}^2 \tag{14}$$

PROOF. To calculate the internal dynamics, set $c = c^*$. Substituting $\dot{c} = \ddot{c} = 0$ in (4) gives an expression for the input, $\tau = \Upsilon_\xi \dot{\xi}^2/\sigma_c$, which is well defined in the neighbourhood of the equilibrium under the hypothesis that $\sigma_c(c^*, \xi^*) \neq 0$. Then, the internal dynamics in the neighbourhood of the equilibrium are given by:

$$\ddot{\xi} = \left(-\Psi_\xi + \Upsilon_\xi \frac{\sigma_\xi}{\sigma_c}\right)\dot{\xi}^2 \tag{15}$$

Since Ψ_ξ, Υ_ξ, σ_ξ, σ_c are functions of only c and ξ and also since $c = c^*$, (15) has the structure $K(\xi)\dot{\xi}^2$.

Theorem 17 *Consider the two-link exclusively-kinetic underactuated nonlinear system (4) with the configuration output, c. If the input-output pair and the configuration of the system are such that* $\sigma_c(c^*, \xi^*) \neq 0$*, then the system is not minimum phase.*

PROOF. $\sigma_c(c^*, \xi^*) \neq 0$ means that the internal dynamics are well defined around the equilibrium point x^*. Consider the function:

$$L(\xi, \dot{\xi}) = \dot{\xi}^2 e^{-2\int_{\xi^*}^{\xi} K(\eta)d\eta}, \tag{16}$$

and its derivative along solutions of (14)

$$\dot{L}(\xi, \dot{\xi}) = 2\dot{\xi}\ddot{\xi}e^{-2\int_{\xi^*}^{\xi} K(\eta)d\eta} + \dot{\xi}^2 e^{-2\int_{\xi^*}^{\xi} K(\eta)d\eta}(-2K(\xi))\dot{\xi}.$$

The following properties hold: (i) $L(\xi^*, 0) = 0$, and (ii) $\dot{L}(\xi, \dot{\xi}) = 0$, $\forall \xi$, $\forall \dot{\xi}$. This implies that L induces level sets on which the trajectories stay. Hence, trajectories starting from the initial conditions, corresponding to a non-zero L, cannot reach the origin. However, the level set corresponding to $L = 0$ is given by $\dot{\xi} = 0$ since the other factor $e^{-2\int_{\xi^*}^{\xi} K(\eta)d\eta}$ never vanishes, Note that the level set has thus dimension one. So, in the neighbourhood of x^* there exist initial conditions with $\dot{\xi} \neq 0$, for which $L(\xi, \dot{\xi}) \neq 0$. For such points x^* is not an attractor since, along the solutions of the system (14), $\dot{L} = 0$. So, the equilibrium point is not asymptotically stable in the Lyapunov sense and the system (4) is nonminimum phase.

7 Examples

In this section, examples of 2-DOF underactuated mechanical systems illustrate the results obtained. Full mathematical analysis will be presented for pendubot and acrobot. Then, the notions of STLC and STLOC will be analyzed on a non-exhaustive list of two-link underactuated mechanical systems.

7.1 The pendubot

The pendubot is a planar 2R robot (an elbow manipulator) with the first joint actuated (Row 1 of Table 1). The inertia matrix corresponding to this system is,

$$D(q) = \begin{bmatrix} J_1 + 2J_3\cos q_2 & J_2 + J_3\cos q_2 \\ J_2 + J_3\cos q_2 & J_2 \end{bmatrix}$$

J_1, J_2, J_3 are constant values depending on the original inertia and geometric parameters. The corresponding dynamics read,

$$\begin{aligned} \ddot{q}_1 &= f_1 + \sigma_1\tau, \qquad \ddot{q}_2 = f_2 + \sigma_2\tau \\ f_1 &= -\frac{J_3\sin q_2}{\Delta D}\left[(J_2 + J_3\cos q_2)\dot{q}_1^2 + 2J_2\dot{q}_1\dot{q}_2 + J_2\dot{q}_2^2\right] \\ f_2 &= -\frac{J_3\sin q_2}{\Delta D}\left[(J_1 + 2J_3\cos q_2)\dot{q}_1^2 + \right. \\ &\quad \left. 2(J_2 + J_3\cos q_2)\dot{q}_1\dot{q}_2 + (J_2 + J_3\cos q_2)\dot{q}_2^2\right] \\ \sigma_1 &= \frac{J_2}{\Delta D}, \qquad \sigma_2 = -\frac{J_2 + J_3\cos q_2}{\Delta D} \end{aligned}$$

7.1.1 Feedback linearization

The only brackets that contain g at most once and do not vanish at the equilibrium point, $q_1 = q_1^*$, $q_2 = q_2^*$, $\dot{q}_1 = 0$ and $\dot{q}_2 = 0$ are $g(x^*)$ and $fg(x^*)$. The Lie brackets are:

$$g(x^*) = \frac{1}{\Delta D}\begin{bmatrix} 0 & 0 & J_2 & -(J_2 + J_3\cos q_2^*) \end{bmatrix}^T$$

$$fg(x^*) = \frac{1}{\Delta D}\left[\begin{array}{cccc} J_2 & -(J_2 + J_3\cos q_2^*) & 0 & 0\end{array}\right]^T$$

Since dim span$\{g, fg, \cdots, f^{n-1}g\} = 2 \neq 4$, the system is not full state feeback linearizable.

7.1.2 STLC

$$gfg = \frac{J_2 J_3^2 \sin(2q_2)}{(\Delta D)^3}\left[\begin{array}{cccc} 0 & 0 & J_2 + J_3\cos q_2 & -(J_1 + 2J_3\cos q_2)\end{array}\right]^T$$

$gfg(x^*) \notin \mathcal{S}^1(x^*)$, $\forall\ q_2^* \neq n\pi/2$, and the pendubot is not STLC at all equilibrium points where $q_2^* \neq n\pi/2$. However, when $q_2^* = (n/2)\pi$, $gfg(q_2^*) = 0$, and the vector which completes the space is $gfgfg(q_2^*)$. Then, the space is completed by brackets with an odd number of g and it can be shown from [13] that the system is STLC at these points. However, these points where the system is STLC form at most a 1-dimensional subspace of the equilibrium manifold.

7.1.3 STLOC

Only the natural choice $c = q_2$ and $\xi = q_1$ is considered ($\sigma_c = \sigma_2 = -\frac{J_2 + J_3\cos q_2}{\Delta D}$ and $\sigma_\xi = \sigma_1 = \frac{J_2}{\Delta D}$). The analysis of the other cases are left to the reader.

Sufficient condition: When $\sigma_c \neq 0, q_2^* \neq \arccos(-J_2/J_3)$, Theorem 13 shows that the system is STLOC.

Necessary condition: The necessary condition supposes $\sigma_c = 0$ and $\frac{\partial \sigma_c}{\partial \xi} \neq 2\sigma_\xi \Upsilon_\xi$, i.e., $q_2^* = \arccos(-J_2/J_3)$ and $\sin q_2^* \neq 0$. These two conditions are met when $J_2 \neq J_3$. In such cases, the system is not STLOC at $q_2^* = \arccos(-J_2/J_3)$.

Gap between Necessary and sufficient conditions: When $J_2 = J_3$, the conditions $\sigma_c = 0$ and $\frac{\partial \sigma_c}{\partial \xi} = 2\sigma_\xi \Upsilon_\xi$ are satisfied at $q_2^* = \pi$. Now, examining the dynamics of q_2 at $q_2^* = \pi$ gives $\ddot{q}_2 = 0$, as long as $q_2^* = \pi$. So, the system can never be pulled out of this configuration which concludes that the system is not STLOC in such a scenario.

7.1.4 Nonminimum-phase property

Case with $\sigma_c \neq 0$: Let $c = q_2$ and $\xi = q_1$. With the assumption $\sigma_c \neq 0$, the point $q_2 = \arccos(-J_2/J_3)$ is eliminated. The internal dynamics of the pendubot (which are well-defined due to the assumption of $\sigma_c \neq 0$) can be computed by imposing the conditions $\dot{q}_2 = \ddot{q}_2 = 0$:

$$\ddot{q}_1 = -\frac{J_3 \sin q_2}{J_2 + J_3\cos q_2}\dot{q}_1^2 \tag{17}$$

The solution of (17) is

$$q_1(t) = q_1(0) + \frac{J_2 + J_3\cos q_2}{J_3 \sin q_2}\log\left(1 + \frac{\dot{q}_1(0) J_3 \sin q_2}{J_2 + J_3\cos q_2} t\right)$$

On the one hand, if $\frac{\dot{q}_1(0)J_3\sin q_2}{J_2+J_3\cos q_2} > 0$, then $q_1(t)$ constantly increases towards infinity as $t \to \infty$. On the other hand, if $\frac{\dot{q}_1(0)J_3\sin q_2}{J_2+J_3\cos q_2} < 0$, the dynamics escapes in finite time $t = -\frac{J_2+J_3\cos q_2}{\dot{q}_1(0)J_3\sin q_2}$. So, the internal dynamics is not asymptotically stable.

Case with $\sigma_c = 0$: Interestingly, the system can be shown to be minimum phase when $\sigma_c = 0$, i.e. at $q_2 = \arccos(-J_2/J_3)$. It can be seen from $\ddot{q}_2 = f_2 + \sigma_2 \tau$, that $\ddot{q}_2 = 0$ implies $\dot{q}_1 = 0$. Next, $q_2^{(3)}$ is proportional to $\tau \dot{q}_1 = 0$, and τ remains undetermined as $\dot{q}_1 = 0$. Considering an extra derivative, $q_2^{(4)}$ is proportional to $\tau^2 = 0$ which forces $\tau = 0$. Since all states and inputs are determined by the output and their derivatives, zero dynamics are absent.

7.2 The acrobot

The acrobot is a planar 2R robot (an elbow manipulator) with the second joint actuated. It has the same inertia matrix as the pendubot. The dynamics are, $\ddot{q}_1 = f_1 + \bar{\sigma}_1\tau$ and $\ddot{q}_2 = f_2 + \bar{\sigma}_2\tau$. The difference lies in $\bar{\sigma}_1 = \frac{-J_2 - J_3\cos q_2}{\Delta D}$ and $\bar{\sigma}_2 = \frac{J_1 + 2J_3\cos q_2}{\Delta D}$, instead of σ_1 and σ_2.

7.2.1 Feedback linearization

The only brackets that contain g at most once and do not vanish at the equilibrium point, $q_1 = q_1^*$, $q_2 = q_2^*$, $\dot{q}_1 = 0$ and $\dot{q}_2 = 0$ are $g(x^*)$ and $fg(x^*)$. The Lie brackets are given by:

$$g(x^*) = \frac{1}{\Delta D}\left[\begin{array}{cccc} 0 & 0 & -(J_2 + J_3\cos q_2^*) & (J_1 + 2J_3\cos q_2^*)\end{array}\right]^T$$
$$fg(x^*) = \frac{1}{\Delta D}\left[\begin{array}{cccc} -(J_2 + J_3\cos q_2^*) & (J_1 + 2J_3\cos q_2^*) & 0 & 0\end{array}\right]^T$$

Since dim span$\{g, fg, \cdots, f^{n-1}g\} = 2 \neq 4$, the system is not full state feeback linearizable.

7.2.2 STLC

$$gfg = \frac{2J_3 \sin q_2 (J_2 + J_3\cos q_2)(J_1 - J_2 + J_3\cos q_2)}{(\Delta D)^2} g$$

It can be seen that $gfg \propto g$. Thus, the system is not accessible at any equilibrium point and thus not STLC. This behaviour is in fact generic to the systems where the second link is actuated. It is interesting to note that though the system is not accessible at any equilibrium point, once the velocities are non-zero, the system becomes accessible.

7.2.3 STLOC

Consider the case $c = q_1$, $\xi = q_2$. Then, $\sigma_c = \bar{\sigma}_1 = -\frac{d_{12}}{\Delta D} = -\frac{J_2 + J_3\cos(q_2)}{\Delta D}$.

Sufficient condition and Necessary condition: As in the case of pendubot, the system is STLOC when $q_2^* \neq \arccos(-J_2/J_3)$ and not STLOC when $q_2^* = \arccos(-J_2/J_3)$ and $J_2 \neq J_3$.

Gap between Necessary and sufficient conditions: When $J_2 = J_3$, the conditions $\sigma_c = 0$ and $\frac{\partial \sigma_c}{\partial \xi} = 2\sigma_\xi \Upsilon_\xi$ are satisfied at $q_2^* = \pi$. For such a case, the fifth order Lie derivatives (with an odd number of g) come to rescue in the functional expansion (11), i.e., $L_{gfggf}c(q_2^*) = -\frac{2}{d_{22}^3 d_{11}} \frac{\partial^2 d_{12}}{\partial q_2^2} \neq 0$ at $q_2^* = \pi$ with $J_2 = J_3$. Since this factor multiplies $\bar{\tau}^3 t^5$ in the expansion, the system is STLOC although $\sigma_c = 0$.

Note that in the gap between necessary and sufficient conditions, pendubot loses STLOC while acrobot does not.

7.2.4 Nonminimum-phase property

If the configuration output is chosen as $c = q_1$, the internal dynamics computed by imposing the conditions $\dot{q}_1 = \ddot{q}_1 = 0$, reads

$$\ddot{q}_2 = -J_2 J_3^2 \sin^2(q_2) \frac{J_1 + 2J_3 \cos(q_2)}{J_2 + J_3 \cos(q_2)} \dot{q}_2^2,$$

which does not have a closed form solution. However, the dynamics has the structure presented in (14) which has been proven to be unstable except at $q_2^* = \arccos(-J_2/J_3)$. As in the case of pendubot, at $q_2^* = \arccos(-J_2/J_3)$ the system can be proven to be minimum phase.

7.3 STLC vs STLOC for a set of two-link underactuated mechanical systems

The concepts of STLC and STLOC will be illustrated on several examples falling into the category of systems for which this paper is devoted. The list of examples (Table 1) is non-exaustive and is chosen to illustrate some fundamental differences that can be present within the class of system considered. Also, a brief sketch of the topographic property of the systems together with inertia matrices are presented in Table 1.

7.3.1 STLC

When the first link is actuated, at almost every equilibrium point x^*, $g(x^*)$, $fg(x^*)$, $gfg(x^*)$ and $fgfg(x^*)$ span a 4 dimensional vector space. Outside the points x^* where $gfg(x^*)$ is zero, the system loses STLC. At the points where $gfg(x^*) = 0$, vectors stemming from Lie brackets of higher order need to be considered. Two cases have been noticed: (i) $gfgfg(x^*)$ adds the necessary dimension, in which case, the system is STLC at this point, and (ii) no higher order brackets can add the neecessary dimension, in which case, the system loses accessibility and thus, is not STLC.

In the first two examples presented in Table 1, at all points x^* where $gfg(x^*) = 0$, $gfgfg(x^*)$ adds the necessary dimension. In the next two cases, $gfg(x^*) = 0$, at $q_2^* = \frac{k\pi}{2}$, $k \in \mathbf{Z}$. For $q_2^* = \frac{(k+1)\pi}{2}$, $gfgfg(x^*)$ adds the necessary dimension leading to STLC. However, when $q_2^* = k\pi$, no higher order brackets can add the dimension and the system loses accessibility. Intuitively, the second link is parallel to the force provided in the first actuator and hence cannot move. The last case is interesting in the sense that, all higher order Lie brackets are identically zero and so the system looses accessibility for all points.

Actuation of the second link is not tabulated since accessibility at equilibirium is lost in all cases, as is the case with acrobot. This inturn implies that the system is not STLC everywhere.

7.3.2 STLOC

The problem of loss of STLOC does not arise in the following cases: (i) $\tau = \tau_1$, $c = q_1$, since $\sigma_c = 0$ implies that the (1,1) element of the inertia matrix is zero, and (ii) $\tau = \tau_2$, $c = q_2$, since $\sigma_c = 0$ implies that the (2,2) element of the inertia matrix is zero. Due to the symmetry of the inertia matrix, the combinations $\tau = \tau_2$, $c = q_1$, and $\tau = \tau_1$, $c = q_2$ lose STLOC at the same points and the results are presented in Table 1.

Four out of five systems considered exhibit the STLOC property for almost all configurations and STLOC is lost at isolated points. However, in the rotary prismatic configuration, STLOC is never lost, while in the perpendicular rotary inverted pendulum, STLOC is lost for all configurations.

It is interesting to compare the two concepts: STLC and STLOC. STLOC is a concept less restrictive than STLC. STLOC is satisfied almost everywhere (except the last example), while STLC is satisfied almost nowhere. At those points where the system is STLC, the system is automatically STLOC.

8 Conclusions

The analysis of two-link exclusively-kinetic underactuated mechanical systems has been undertaken. The results can be summarized as follows:

- The systems of this class are not feedback linearizable.
- STLC is not verified except for a few configurations.
- STLOC is verified in most configurations.

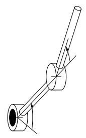
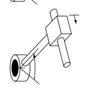
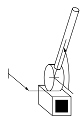
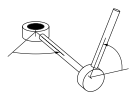
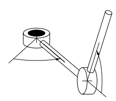
System	Illustration	D(q)	STLC at	Not STLOC at
Pendubot		$\begin{bmatrix} J_1+2J_3\cos q_2 & J_2+J_3\cos q_2 \\ J_2+J_3\cos q_2 & J_2 \end{bmatrix}$	$q_2 = \frac{k\pi}{2}$	$q_2 = \arccos(-\frac{J_2}{J_3})$
Rotary-Prismatic		$\begin{bmatrix} J_1+J_3 q_2^2 & -J_4 \\ -J_4 & J_2 \end{bmatrix}$	$q_2 = 0$	–
Inverted Pendulum		$\begin{bmatrix} J_1 & -J_3\sin q_2 \\ -J_3\sin q_2 & J_2 \end{bmatrix}$	$q_2 = \frac{(k+1)\pi}{2}$	$q_2 = k\pi$
Rotational Inv. Pend.		$\begin{bmatrix} J_1+J_3\cos(2q_2) & -J_4\sin q_2 \\ -J_4\sin q_2 & J_2 \end{bmatrix}$	$q_2 = \frac{(k+1)\pi}{2}$	$q_2 = k\pi$
$\perp$-Rot. Inv. Pend.		$\begin{bmatrix} J_1+J_3\cos q_2+J_4\cos(2q_2) & 0 \\ 0 & J_2 \end{bmatrix}$	–	$\forall q_2$

Table 1
STLC and minimum phase analysis of two-link under-actuated mechanical systems. The first link is actuated $\tau = \tau_1$ and the angle of the second link is chosen as the output $c = q_2$. $k \in \mathbf{Z}$.

- The system is NMP in most configurations.

The examples of pendubot and acrobot are discussed in detail and a table of different two-link underactuated systems is presented to illustrate the difference between STLC and STLOC.

Generalization of these results to more than two links and single input is envisaged. Also, the influence of gravity and friction needs to be considered.

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