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Bavula, V. V. (4-SHEF-PM)

The inversion formulae for automorphisms of polynomial algebras and rings of differential operators in prime characteristic. (English summary)

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Inversion formulas for a class of algebras—over a field of prime characteristic—are derived. Among such algebras, the paper considers (i) the polynomial algebra $P_n := K[x_1, x_2, \ldots, x_n]$; (ii) the ring of differential operators $\mathcal{D}(P_n)$ on P_n, i.e., $\mathcal{D}(P_n) \otimes P_n$; (iii) the n-th Weyl algebra A_n; (iv) the algebra $P_n \otimes A_m$; (v) the power series algebra $K[[x_1, \ldots, x_n]]$; (vi) $T_{k_1,\ldots,k_n} \otimes P_m$, where $T_{k_1,\ldots,k_n}$ is the subalgebra of $\mathcal{D}(P_m)$ generated by P_n and the higher derivations $\partial_i^{[j]}$, $0 \leq j < p^{k_i}$, $i = 1, \ldots, n$, where $k_1, \ldots, k_n \in \mathbb{N}$.

The general idea for finding the inversion formula is the following. Let A be an algebra over the field K, σ an automorphism over A, and $\{x^\alpha\}$ a K-basis of A. The identity of the algebra is first decomposed as $\mathrm{id}_A(\cdot) = \sum \lambda_{\alpha,y_\alpha}(\cdot)x^\alpha$, where $\lambda_{\alpha,y_\alpha}$ are algebraic maps. Both this decomposition and the existence of the inverse are assumed to exist for σ. Applying $\sigma(\cdot)$ to the identity map also has the presentation $\mathrm{id}_A(\cdot) = \sum \lambda_{\alpha,\sigma(y_\alpha)}(\cdot)\sigma(x^\alpha)$. Then because $\lambda_{\alpha,\sigma(y_\alpha)}(A) \subseteq K$, one simply applies $\sigma^{-1}(\cdot)$ to obtain the inverse expressed as $\sigma^{-1}(\cdot) = \sum \lambda_{\alpha,\sigma(y_\alpha)}(\cdot)x_\alpha$. The whole difficulty is to find suitable maps $\lambda_{\alpha,\sigma(y_\alpha)}(\cdot)$.

So as to be more specific, some definitions and notation are required. Let δ be a K-derivation of an algebra A over an arbitrary field K. A finite or infinite sequence $x = \{x^{[i]},\ 0 \leq i \leq l-1\}$ of elements in A where $x^{[0]} = 1$ is called an iterative sequence of length l if $x^{[i]}x^{[j]} = \binom{i+j}{i}x^{[i+j]}$, $0 \leq i, j \leq i-1$, $i+j \leq l-1$. An iterative δ descent is such a sequence for which $\delta(x^{[i]}) = x^{[i-1]}$, $0 \leq i \leq l-1$, $x^{[-1]} = 0$. Whenever δ is nilpotent, i.e., $\delta^l = 0$ for some $l \geq 2$, two K-linear maps from A to A can be constructed starting from an iterative δ sequence $\{x^{[i]},\ 0 \leq i < l\}$ in the following way: $\varphi := \sum_{i=0}^{l-1}(-1)^i x^{[i]}\delta^i(\cdot)$ and $\psi := \sum_{i=0}^{l-1}(-1)^i\delta^i(\cdot)x^{[i]}$. These maps are projection maps onto the kernel A^δ of δ; that is, if $c \in A$ is written as $c = a + b$ with $a \in A^\delta$ and $b \in A_+$, which is always possible since $A = A^\delta \oplus A_+$ with $A_+ := \bigoplus_{i=1}^{l-1} x^{[i]}A^\delta$, then $\psi(c) = \varphi(c) = a$.

In the case of an automorphism σ that preserves the ring of invariants in the sense that $\sigma(A^\delta) = A^\delta$, the following concepts are required. Both a non-empty well-ordered set I and a set of commuting locally nilpotent K-derivations $\delta := \delta_i$, $i \in I$, are given. Suppose that for each $i \in I$ there exists an iterative δ_i-descent $\{x_i^{[j]}\}$ of maximal length such that $\{x_i^{[j]}\} \subseteq \bigcap_{i \neq k \in I} A^{\delta_k}$. Define (i) $\sigma_\delta := \sigma|_{A_\delta}$; (ii) the twisted derivations δ' as $\{\delta'_i := \sigma\delta_i\sigma^{-1},\ i \in I\}$; and (iii) the images of the iterative descents $x'_i := \{x_i'^{[j]} := \sigma(x_i^{[j]})\}$, $i \in I$. The inversion formula is shown to be

$$\sigma^{-1}(a) = \sum_{\alpha \in E} x^{[\alpha]}\sigma_\delta^{-1}\varphi_\sigma(\delta'^\alpha(a)) = \sum_{\alpha \in E} \sigma_\delta^{-1}\psi_\sigma(\delta'^\alpha(a))x^{[\alpha]}.$$

The result for $\sigma \in \mathrm{Aut}_K(\mathcal{D}(K[x_1, \ldots, x_n]))$ is more involved, but nevertheless still rests on suitable locally nilpotent derivations and their nil algebras. $\mathcal{D}(K[x_1, \ldots, x_n])$ is the ring of differential operators on the polynomial algebra $K[x_1, \dots, x_n]$. This algebra is a K-algebra generated by the elements $x_1, \dots, x_n$ and commuting higher derivations $\partial_i^{[k]} := \frac{\delta_i^k}{k!}$, $i = 1, \dots, n$ and $k \geq 1$, that satisfy the defining relations $[x_i, x_j] = [\partial_i^{[k]}, \partial_j^{[l]}] = 0$, $\partial_i^{[k]}\partial_i^{[l]} = \binom{k+l}{k}\partial_i^{[k+l]}$, $[\partial_i^{[k]}, x_j] = \delta_{ij}\partial_i^{[k-1]}$ for all $i, j = 1, \dots, n$ and $k, l \geq 1$ where δ_{ij} is the Kronecker delta and $\partial_i^{[0]} := 1$, $\partial_i^{[1]} = \frac{\partial}{\partial x_i}$. Given two elements x_i, x_j, define the inner derivation as $[x_i, x_j] = (\mathrm{ad}\, x_i)(x_j) := x_i x_j - x_j x_i$. The key projection maps are $\varphi_i = \sum_{k \geq 0} \partial_i^{[k]}(\mathrm{ad}\, x_i)^k$, $\psi_i = \sum_{k \geq 0}(\mathrm{ad}\, x_i)^k \partial_i^{[k]}$, $i = 1, \dots n$, which are used to infer the validity of the inversion formula

$$\sigma^{-1}(a) = \sum_{\alpha, \beta \in \mathbb{N}^n} (-1)^{|\alpha|} \varphi_\sigma(\delta'^\beta(a)\partial'^{[\alpha]})\partial^{[\beta]} x^\alpha =$$

$$\sum_{\alpha, \beta \in \mathbb{N}^n} \psi_\sigma(\partial'^{[\alpha]}\delta'^\beta(a)) x^\alpha \partial^{[\beta]},$$

where $\delta'^\beta := \prod_{i=1}^n (-\mathrm{ad}\, x_i')^{\beta_i}$, the primed quantities being $x_i' := \sigma(x_i)$ and $\partial_i'^{[k]} := \sigma(\partial_i^{[k]})$.

Reviewed by *Philippe A. Müllhaupt*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.