On a problem in relation with the values of the argument of the Riemann zeta function in the neighborhood of points where zeta is large

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December 2011

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Keywords: Riemann zeta function, Distribution of zeros.

Abstract

The aim of this research is to establish a relation between the derivatives of Hardy's Z function and the argument of the Riemann zeta function in the neighborhood of points where $|Z|$ reaches a large maximum. In this paper, we make a step toward this goal by solving a problem of the same nature.

Introduction

To situate the problem we address here, we recall some classical results on the zeros of the Riemann zeta function.
We denote as usual by Z the Hardy function whose real zeros coincide with the zeros of ζ located on the line of real part $\frac{1}{2}$. If the Riemann hypothesis is true, what we assume from now on, then the number of zeros of Z in the interval $]0,t]$ is given by

$$N(t) = \frac{t}{2\pi}\log\left(\frac{t}{2\pi e}\right) + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right) \tag{1}$$

where $S(t) = \dfrac{1}{\pi}\arg\zeta(\dfrac{1}{2}+it)$ if t is not a zero of Z and $\arg\zeta(\frac{1}{2}+it)$ is defined by continuous variation along the straight lines joining 2, $2+it$ and $\frac{1}{2}+it$ starting with the initial value $\arg\zeta(2)=0$. If t is a zero of Z we set $S(t) = \lim_{\epsilon\to 0_+} S(t+\epsilon)$. The function S is called the argument of zeta, it satisfies $S(t) = O(\dfrac{\log t}{\log\log t})$ [6].
Given a point $T > 2\pi$ at which $|Z|$ reaches a maximum satisfying $|Z(T)| = \max_{0\leqslant t\leqslant T}|Z(t)|$, we denote by γ_k the real zeros of Hardy's function numbered so that $\cdots \leqslant \gamma_{-2} \leqslant \gamma_{-1} < T < \gamma_1 \leqslant \gamma_2 \cdots$. From (1) we infer that

$$|\gamma_{\pm k} - T| \leqslant (k-1)\frac{2\pi}{\log\frac{T}{2\pi}} + O(\frac{1}{\log\log T}) \quad \text{for } k = 1, 2, \cdots, l$$

where $l = \lfloor T^{\frac{1}{2}} \rfloor$.

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As a result the function g defined by

$$g(t) = \frac{Z(T + \dfrac{2t}{\log \frac{T}{2\pi}})}{Z(T)}$$

satisfies $g(0) = 1$ and its zeros $x_{\pm k} = (\gamma_{\pm k} - T)\dfrac{\log \frac{T}{2\pi}}{2}$ are such that

$$| x_{\pm k} | \leqslant (k-1)\,\pi + O(\frac{\log T}{\log\log T}) \quad \text{for } k = 1, 2, \cdots, l.$$

Numerical investigations, using in particular values of T given by [4], suggest there exists $0 < c_T < 1$ such that $| g^{(j)} | \leqslant c_T^j$ on the interval $[x_{-l}, x_l]$ for $j \leqslant \frac{1}{2}\log T$ [1]. The objective of this research is to link the maximum of S in a neighborood of T and the constant c_T. In this paper we solve a similar problem, demonstrating the feasibility of this project.

Our method could play a role in a conditional proof of the existence of arbitrary large normalized gaps between consecutive zeros of the Riemann zeta function (see Remarks 1 b) and c)).

This work stems from an observation of A.Ivič [3] about the values of the derivatives of Z in a neighborood of points where $|Z|$ reaches a large value.

The notations used in this paper are standard: we set $(x)_+ = \max(x, 0)$ and $(x)_- = \min(x, 0)$. Bernoulli and Euler polynomials are denoted respectively by $B_n(x)$ and $E_n(x)$. We recall that these are related to Bernoulli polynomials by

$$E_n(x) = \frac{2^{n+1}}{n+1}(B_{n+1}(\frac{1+x}{2}) - B_{n+1}(\frac{x}{2})).$$

Main Result

The theorem that we state here admits generalizations which we will discuss in the conclusion of this paper.

Theorem

Let $f \in C^\infty[-x_l, x_l]$ be an even function satisfying $f(0) = 1$ and vanishing at x_k where $0 < x_1 < \cdots < x_l$. We assume moreover that there exists a constant $0 < c < 1$ such that $|f^{(2j-1)}(x_l)| \leqslant c^{2j-1}$ and $f^{(2j)} = O(c^{2j})$ for all $j \in \mathbb{N}^*$ and that there exists $s > 0$ such that

$$x_k \leqslant (k-1)\pi + s \quad \text{for} \quad k = 1, 2, \cdots, l-1 \quad \text{and} \quad x_l = (l-1)\pi + s\,.$$

Further, for $n \in \mathbb{N}^*$ such that $c(l-1) < n \leqslant l$, let $f_c^{(l,n)}$ be the function defined on the interval $[0, (\frac{n}{c} - (l-1))\pi[$ by [2]

$$f_c^{(l,n)}(s) = 1 + \lim_{t \to s_+} \frac{\prod\limits_{1 \leqslant j \leqslant n} \sin^2(\pi \dfrac{x_j^*(t)}{2x_l^*(t)})}{\sin(c\, x_l^*(t))} \left(\sum_{k=0}^{n} \frac{\cos(c\, x_k^*(t))}{\prod\limits_{\substack{0 \leqslant j \leqslant n \\ j \neq k}} \left(\sin^2(\pi \dfrac{x_k^*(t)}{2x_l^*(t)}) - \sin^2(\pi \dfrac{x_j^*(t)}{2x_l^*(t)}) \right)} \right)$$

where $x_0^*(t) = 0$ and $x_k^*(t) = (k-1)\pi + t$ for $k = 1, \cdots, l$.

Then

$$s \geq s^*(c, l) \overset{def}{=} \max_{c(l-1) < n \leqslant l} s^{(n)}(c, l)$$

where $s^{(n)}(c, l)$ is the unique solution of equation $f_c^{(l,n)}(s) = 0$.

[1]This choice follows from a formula of Lavrik[5] which allows the computation of an approximation of $Z^{(j)}(t)$ for $j \leqslant \frac{1}{2}\log t$.

[2]We use a limit in the definition of $f_c^{(l,n)}$ to remove the singularities.

Before addressing the proof of this theorem, we make some preliminary observations.

Remarks 1

a) The case $l = 1$ leads to the equation $1 - \tan(\frac{cs}{2}) = 0$ which gives the optimal lower bound $s^*(c, 1) = \frac{\pi}{2c}$, this latter is reached for the function $f(x) = \cos(c\,x)$. We will show that this result holds under weaker assumptions.

b) Figure 1 gives the values of the lower bound $s^*(c, l)$ for the constant $c = 0.95$. It suggests, with other computations, a linear increasing of $s^*(c, l)$ with respect to l.

c) The proof of this theorem shows that if s is close to s^* then x_1 is close to s^*. This observation provides an additional argument for the existence of large normalized gaps between two consecutive zeros of the zeta function.

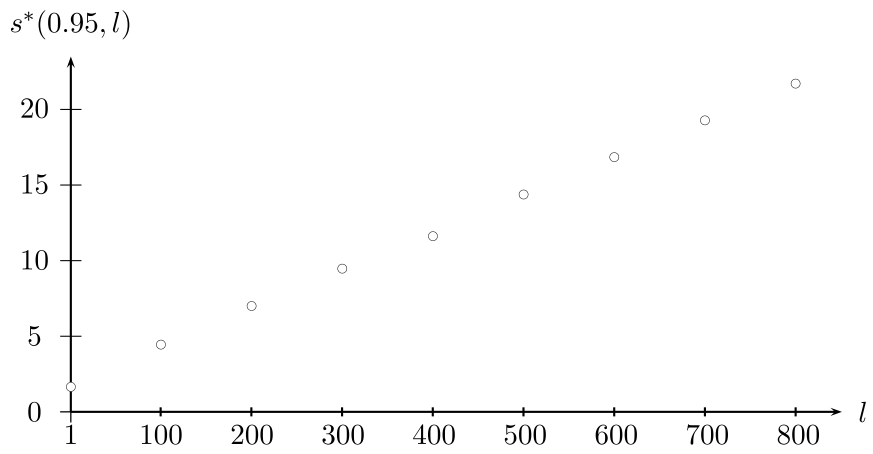

Figure 1: Values of $s^*(0.95, l)$ for $l = 1, 100, \cdots, 800$.

A fundamental identity

We first establish a relation between the value of an even function $f \in C^\infty[-x_l, x_l]$ at 0, the zeros of f and the values of its derivatives of odd order on the boundaries of the interval. This relation, together with the expressions given in Lemma 2, is the fundamental identity.

Lemma 1
Let $0 = x_0 < x_1 < \cdots < x_l$ and $f \in C^\infty[-x_l, x_l]$ be an even function vanishing at $x_1, x_2, \cdots, x_l$. For $m \geq 1$ we introduce the function Ψ_{2m-1} defined (up to an additive constant) by

$$\Psi_{2m-1}(x) = p_{2m-2}(x) + \sum_{k=0}^{l-1} \mu_k \frac{(x - x_k)_+^{2m-1}}{(2m-1)!}$$

where $\mu_0 = 1, \mu_1, \cdots, \mu_{l-1} \in \mathbb{R}$ and p_{2m-2} is a polynomial of degree at most equal to $2m - 2$ which, as $m \geq 2$, is such that $\Psi_{2m-1}^{(2n-1)}(0) = 0$ and $\Psi_{2m-1}^{(2n-1)}(x_l) = 0$ for $n = 1, 2, \cdots, m - 1$.
Then we have the identity

$$f(0) = -\sum_{k=1}^{m} f^{(2k-1)}(x_l)\Psi_{2m-1}^{(2m-2k)}(x_l) + \int_0^{x_l} f^{(2m)}(x)\Psi_{2m-1}(x)\,dx. \qquad (2)$$

Proof
As the function $\Psi_{2m-1}^{(2m-1)}$ is constant on each interval $]x_k, x_{k+1}[$ and this constant is equal to 1 on the interval $]x_0, x_1[$ we have

$$\int_0^{x_1} f'(x)\Psi_{2m-1}^{(2m-1)}(x)\,dx = -f(0) \quad \text{and} \quad \int_{x_k}^{x_{k+1}} f'(x)\Psi_{2m-1}^{(2m-1)}(x)\,dx = 0 \quad \text{for } k = 1, 2, \ldots, l-1.$$

Summing these equalities we get

$$f(0) = -\int_0^{x_l} f'(x)\Psi_{2m-1}^{(2m-1)}(x)\,dx$$

and we complete the proof by integrating $2m-1$ times the right-hand side by parts, taking into account that the derivatives of odd order of f vanish at 0 and that $\Psi_{2m-1}^{(2n-1)}(0) = \Psi_{2m-1}^{(2n-1)}(x_l) = 0$ for $n = 1, 2, \cdots, m-1$. $\square$

To continue we need the explicit expression of the function Ψ_{2m-1}. This is the result of the following lemma.

Lemma 2
Let $q_{k,2m-2}$, $\theta_{k,2m-1}$ and Ψ_{2m-1} be the functions defined[3] for $0 \leqslant k \leqslant l$ and $m \geq 1$ by

$$q_{k,2m-2}(x) = \frac{2^{2m-1}x_l^{2m-1}}{(2m)!}(2B_{2m}(\frac{1}{2} + \frac{x}{2x_l}) - B_{2m}(\frac{x_k + x}{2x_l}) - B_{2m}(\frac{x_k - x}{2x_l})), \tag{3}$$

$$\theta_{k,2m-1}(x) = q_{k,2m-2}(x) + \frac{(x - x_k)_+^{2m-1}}{(2m-1)!} \tag{4}$$

and

$$\Psi_{2m-1}(x) = \sum_{k=0}^{l} \mu_k \theta_{k,2m-1}(x) \tag{5}$$

where $\mu_0 = 1$. Then

a) The polynomial $q_{k,2m-2}$ is an even polynomial of degree at most equal to $2m-2$.

b) $\theta_{k,2m-1}^{(2n-1)}(0) = 0$ and $\theta_{k,2m-1}^{(2n-1)}(x_l) = 0$ for $n = 1, 2, \ldots, m-1$.

c) The function Ψ_{2m-1} satisfies the conditions stated in Lemma 1.

d) $\Psi_{2m-1}^{(2m-2k)} = \Psi_{2k-1}$ for $k = 1, 2, \ldots, m$.

Proof

a) By definition, the polynomial $q_{k,2m-2}$ is of degree at most equal to $2m-1$.
The identity $B_{2m}(\frac{1}{2} + x) = B_{2m}(\frac{1}{2} - x)$ shows that $q_{k,2m-2}(x) = q_{k,2m-2}(-x)$. Hence $q_{k,2m-2}$ is an even polynomial of degree at most equal to $2m-2$.

b) The fact that the derivatives of order $1, 3, \ldots, 2m-3$ of $\theta_{k,2m-1}^{(2n-1)}$ vanish at 0 is a consequence of a).
The identity $B_n(x) = B_n(x+1) - nx^{n-1}$ allows us to rewrite $\theta_{k,2m-1}$ as

$$\theta_{k,2m-1}(x) = \frac{2^{2m-1}x_l^{2m-1}}{(2m)!}(2B_{2m}(\frac{1}{2} + \frac{x}{2x_l}) - B_{2m}(\frac{x_k + x}{2x_l}) - B_{2m}(\frac{x_k - x}{2x_l} + 1)) - \frac{(x - x_k)_-^{2m-1}}{(2m-1)!}. \tag{6}$$

We check that the derivatives of order $1, 3, \ldots, 2m-3$ of $\theta_{k,2m-1}$ vanish at x_l by using the properties $B_n'(x) = nB_{n-1}(x)$ and $B_{2n+1}(1) = 0$ for $n = 1, 2, \ldots$

[3] It is convenient to define $\theta_{l,2m-1}$ altough $\theta_{l,2m-1}$ is null on $[x_0, x_l]$.

c) This is an immediate consequence of a) and b).

d) This assertion follows from the relation

$$\theta_{k,2m-1}^{(2j)}(x) = q_{k,2m-2}^{(2j)}(x) + \frac{(x-x_k)_+^{2m-2j-1}}{(2m-2j-1)!} = q_{k,2m-2j-2}(x) + \frac{(x-x_k)_+^{2m-2j-1}}{(2m-2j-1)!} = \theta_{k,2m-2j-1}(x)$$

valid for $j = 0, 1, \cdots, m-1$. □

Remark 2
To compute the polynomial $q_{k,2m-2}$ we have set $q_{k,2m-2}(x) = \sum_{k=0}^{m-1} \frac{a_{2k}}{(2k)!} x^{2k}$ and we have solved the triangular system satisfied by the coefficients $a_2, a_4, \cdots, a_{2m-2}$. Bernoulli numbers appear in the inverse of the system's matrix. To see this we note that the rows of the matrix A of the system contain the Taylor expansion of $\sinh(x_l)$. We verify that the rows of the matrix A^{-1} contain the Taylor expansion of the function

$$\frac{1}{\sinh(x_l)} = \sum_{k=0}^{\infty} \frac{2-2^{2k}}{(2k)!} B_{2k} x_l^{2k-1}$$

by observing that the product of the i-th row of A by the j-th column of A^{-1} where $j \geq i$ gives the term of degree $2(j-i)$ of the Taylor expansion of the function $\sinh(x_l)\frac{1}{\sinh(x_l)} = 1$; this product is therefore equal to δ_{ij}. We omit details of the calculations leading to (3) since the verification of Lemma 2 is easy.

From lemmas above we derive a first result for the case $l = 1$.

Proposition
Let $f \in C^\infty[-x_1, x_1]$ be an even function satisfying $f(0) = 1$ and vanishing at x_1. We further assume that there exists a constant $c > 0$ such that $|f^{(2n-1)}(x_1)| \leqslant c^{2n-1}$ and $f^{(2n)} = O((2c)^{2n})$ for all $n \in \mathbb{N}^*$. Then $x_1 \geq \frac{\pi}{2c}$.

Proof
Let f be an even function satisfying the assumptions of the proposition for some x_1 which is assumed to belong to the interval $]0, \frac{\pi}{2c}[$. This will lead to a contradiction.
Using (6) and relations $B_{2m}(1-x) = B_{2m}(x)$ and $E_{2m-1}(2x) = \frac{2^{2m}}{2m}(B_{2m}(\frac{1}{2}+x) - B_{2m}(x))$ we can check that functionΨ_{2m-1} defined by (5) satisfies

$$\Psi_{2m-1}(x) = E_{2m-1}(\frac{x}{x_1})\frac{x_1^{2m-1}}{(2m-1)!}$$

and by Lemma 2 d), relation (2) writes

$$f(0) = -\sum_{k=1}^{m} f^{(2k-1)}(x_1)E_{2k-1}(1)\frac{x_1^{2k-1}}{(2k-1)!} + \int_0^{x1} f^{(2m)}(x)E_{2m-1}(\frac{x}{x_1})\frac{x_1^{2m-1}}{(2m-1)!}\,dx\,. \tag{7}$$

Thanks to properties

$$\max_{0\leqslant x\leqslant x_1} \left|E_{2k-1}(\frac{x}{x_1})\frac{x_1^{2k-1}}{(2k-1)!}\right| = \left|E_{2k-1}(1)\right|\frac{x_1^{2k-1}}{(2k-1)!} = \frac{2(2^{2k}-1)}{(2k)!}|B_{2k}|x_1^{2k-1} = 0(\frac{x_1^{2k-1}}{\pi^{2k-1}})$$

we can let m go to infinity in (7) since $x_1 \in]0, \frac{\pi}{2c}[$ and deduce the bound

$$f(0) \leqslant \sum_{k=1}^{\infty} |E_{2k-1}(1)|\frac{(cx_1)^{2k-1}}{(2k-1)!} = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{(2k)!}|B_{2k}|\,(\frac{cx_1}{2})^{2k-1} = \tan(\frac{cx_1}{2}) < 1.$$

This is a contradiction because $f(0) = 1$. □

Our theorem is a generalization of the above proposition to even functions vanishing at $x_1 < x_2 < \cdots < x_l$. In its proof we will let m go to infinity in (2) for a sufficiently large x_l and this will impose conditions on the coefficients μ_k.

The choice of coefficients μ_k

As shown in the proof of Lemma 4, the functions $\theta_{k,2m-1}$ are $O((\frac{x_l}{\pi})^{2m-1})$ and usually the same holds for the function Ψ_{2m-1}. If l is large then $c\,x_l > \pi$ and getting a non trivial lower bound of s requires $\Psi_{2m-1} = O_l((\frac{x_l}{n\pi})^{2m-1})$ for some n such that $c\,x_l < n\,\pi$. This is possible for a choice of μ_k that follows from classical results on divided differences that we recall in the next lemma.

Lemma 3
Let $t_0 < t_l$ be two fixed numbers, $f \in C[t_0, t_l]$, $1 \leqslant n \leqslant l$ and let moreover g_n be the function defined by

$$g_n(t_1, \cdots, t_n) = \sum_{k=0}^{n} \frac{f(t_k)}{\prod\limits_{\substack{0\leqslant j\leqslant n\\ j\neq k}} (t_k - t_j)} \quad \text{for } t_0 < t_1 < \cdots < t_n < t_l$$

if $1 \leqslant n \leqslant l-1$ and by

$$g_n(t_1, \cdots, t_{l-1}) = \sum_{k=0}^{l} \frac{f(t_k)}{\prod\limits_{\substack{0\leqslant j\leqslant l\\ j\neq k}} (t_k - t_j)} \quad \text{for } t_0 < t_1 < \cdots < t_{l-1} < t_l$$

if $n = l$. Then g_n has the following properties:

a) If $n \leqslant l-1$ and if $f \in C^n[t_0, t_l[$ then there exists $\xi = \xi(t_1, \cdots, t_n) \in]t_0, t_n[$ such that

$$g_n(t_1, \cdots, t_n) = \frac{f^{(n)}(\xi)}{n!}.$$

Moreover if $f \in C^{n+1}[t_0, t_l[$ is such that $f^{(n+1)} > 0$ on $[t_0, t_l[$ then $\dfrac{\partial}{\partial t_i} g_n > 0$ for $i = 1, \cdots, n$.

b) Assertions stated in a) hold when $n = l$ with the modification $\dfrac{\partial}{\partial t_i} g_n > 0$ for $i = 1, \cdots, l-1$.

Proof
When $1 \leqslant n \leqslant l-1$ assertions are consequences of the representation formula

$$g_n(t_1, \cdots, t_n) = \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} f^{(n)}\,(t_0 + \sum_{k=1}^{n} \tau_k (t_k - t_{k-1}))\, d\tau_n\,.$$

Generalization to the case $n = l$ is classical. $\square$

Lemma 4
Let $0 = x_0 < x_1 < \cdots < x_l$ and let $1 \leqslant n \leqslant l$ and $\mu^*_{n,0},\ \mu^*_{n,1}, \cdots,\ \mu^*_{n,l}$ be the numbers defined by

$$\mu^*_{n,k} = \begin{cases} (-1)^n \left(\prod\limits_{1\leqslant j\leqslant n} \sin^2(\pi \dfrac{x_j}{2x_l}) \right) \dfrac{1}{\prod\limits_{\substack{0\leqslant j\leqslant n\\ j\neq k}} \left(\sin^2(\pi \dfrac{x_k}{2x_l}) - \sin^2(\pi \dfrac{x_j}{2x_l}) \right)} & \text{for } k = 0, 1, \cdots, n \\ 0 \quad \text{for } k = n+1, \cdots, l. \end{cases}$$

Then $\mu^*_{n,0} = 1$ and the function $\Psi^*_{n,2m-1}$ defined on the interval $[0, x_l]$ by

$$\Psi^*_{n,2m-1}(x) = \sum_{k=0}^{l} \mu^*_{n,k} \theta_{k,2m-1}(x)$$

where $\theta_{k,2m-1}$ are given by (4) is a $O_l((\frac{x_l}{n\pi})^{2m-1})$.

Proof
We use the Fourier series expansion

$$B_{2m}(x) = (-1)^{m+1} 2((2m)!) \sum_{j=1}^{\infty} \frac{1}{(2j\pi)^{2m}} \cos(2j\pi x) \quad \text{for } x \in [0,1]. \tag{8}$$

When $x \in [0, x_k]$, the arguments of Bernoulli polynomials which appear in (4) are located in the interval $[0,1]$ and formula (8) applies. Similarly, when $x \in [x_k, x_l]$, formula (8) applies to relation (6). Both calculations give the same result and the use of identities $\cos\alpha + \cos\beta = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$ and $\cos\alpha = 1 - 2\sin^2(\frac{\alpha}{2})$ leads to the expression

$$\theta_{k,2m-1}(x) = (-1)^{m+1} \, 4 \, \frac{x_l^{2m-1}}{\pi^{2m}} \sum_{j=1}^{\infty} \frac{a_{j\,,\,k}}{j^{2m}} \cos(j\pi \frac{x}{x_l}) \quad \text{for } x \in [0, x_l]$$

where

$$a_{j\,,\,k} = \sin^2(j\pi \frac{x_k}{2x_l}) - \frac{(-1)^{j+1}+1}{2}.$$

It follows that

$$\begin{aligned}
\Psi^*_{n,2m-1}(x) &= (-1)^{m+1} \, 4 \, \frac{x_l^{2m-1}}{\pi^{2m}} \sum_{j=1}^{\infty} \sum_{k=0}^{n} \frac{\mu^*_{n,k} a_{j\,,\,k}}{j^{2m}} \cos(j\pi \frac{x}{x_l}) \\
&= (-1)^{m+1} \, 4 \, \frac{x_l^{2m-1}}{\pi^{2m}} \left(\sum_{j=1}^{n-1} \sum_{k=0}^{n} \frac{\mu^*_{n,k} a_{j\,,\,k}}{j^{2m}} \cos(j\pi \frac{x}{x_l}) + \sum_{j=n}^{\infty} \sum_{k=0}^{n} \frac{\mu^*_{n,k} a_{j\,,\,k}}{j^{2m}} \cos(j\pi \frac{x}{x_l}) \right).
\end{aligned}$$

From Lemma 3, if f is a polynomial of degree at most equal to $n-1$ then $\sum_{k=0}^{n} \mu^*_{n,k} \, f(\sin^2(\pi \frac{x_k}{2x_l})) = 0$.
As $\sin^2(j\pi \frac{x}{2x_l})$ is a polynomial of degree j in $\sin^2(\pi \frac{x}{2x_l})$, we have $\sum_{k=0}^{n} \mu^*_{n,k} \, a_{j\,,\,k} = 0$ for $j = 1, \cdots, n-1$ and this implies that $\Psi^*_{n,2m-1} = O_l((\frac{x_l}{n\pi})^{2m-1})$. □

Thanks to the bound we get in Lemma 4 we will be able to let m go to infinity in relation (2). As in the proposition concerning the case $l = 1$ we will identify the limit function.

A property of Bernoulli polynomials

The following inequality is essential in demonstrating the properties of functions $\Psi^*_{n,2m-1}$.

Lemme 5
For all $m, k \in \mathbb{N}^*$ we have the inequality

$$(-1)^{m+1} \frac{d^k}{dx^k} B_{2m}\left(\frac{1}{2} + \frac{1}{\pi} \text{Arcsin}\sqrt{x} \right) > 0 \quad \text{for } x \in [0,1[\,.$$

The proof of Lemma 5 requires two technical results given in Sublemmas 1 and 2.

Sublemma 1
For all $k \in \mathbb{N}$ we have the Taylor expansion

$$(\operatorname{Arcsin} x)^{2k} = \sum_{l=0}^{\infty} \frac{(2k)!}{(2l)!} 2^{2l-2k}\, b_{k\,,\,l}\, x^{2l} \qquad \text{for } x \in [-1,1]$$

where $b_{k\,,\,l}$ are integers defined recursively by

$$\begin{cases} b_{0\,,\,0} = 1 \text{ and } b_{k\,,\,0} \;=\; b_{0\,,\,l} = 0 & \text{for } k,l \geq 1 \\ b_{k+1\,,\,l+1} \;=\; b_{k\,,\,l} \;+\; l^2 b_{k+1\,,\,l} & \text{for } k,l \geq 0\,. \end{cases}$$

Proof
We note first that the functions $f_{2k}(x) \stackrel{def}{=} (\operatorname{Arcsin} x)^{2k}$ satisfy

$$(1-x^2) f''_{2k+2}(x) - x\, f'_{2k+2}(x) - (2k+2)(2k+1) f_{2k}(x) = 0 \qquad \text{for } x \in \,]-1,1[.$$

From the definition of f_{2k} and the above equality it follows that numbers $c_{k\,,\,l}$ defined by

$$f_{2k}(x) = \sum_{l=0}^{\infty} c_{k\,,\,l}\, x^{2l} \qquad \text{for} \qquad x \in [-1,1]$$

are uniquely determined by the recurrence relations

$$\begin{cases} c_{0\,,\,0} \;=\; 1 \text{ and } c_{k\,,\,0} \;=\; c_{0\,,\,l} \;=\; 0 \quad \text{for } k,\, l \geq 1 \\ (2l+2)(2l+1) c_{k+1\,,\,l+1} - 4l^2 c_{k+1\,,\,l} - (2k+2)(2k+1) c_{k\,,\,l} = 0 \;\; \text{for } k,\, l \geq 0\,. \end{cases}$$

A simple check shows that $c_{k\,,\,l} = \dfrac{(2k)!}{(2l)!} 2^{2l-2k} b_{k\,,\,l}$. $\square$

Sublemma 2
Let $b_{k\,,\,l}$ be the numbers defined in Sublemma 1. Then

$$\lim_{l\to\infty} \frac{b_{k\,,\,l}}{((l-1)!)^2} = \frac{\pi^{2k-2}}{(2k-1)!} \quad \text{for all } k \geq 1. \tag{9}$$

Proof
From the definition of numbers $b_{k\,,\,l}$ we infer that $b_{1\,,\,l} = ((l-1)!)^2$ for $l \geq 1$. Thus relation (9) is trivially true for $k = 1$. We then assume $k \geq 2$. As $b_{j\,,\,1} = 0$ for $j \geq 2$ the numbers $d_{j\,,\,l}$ defined for $j,\, l \geq 1$ by $d_{j\,,\,l} = \dfrac{b_{j\,,\,l}}{((l-1)!)^2}$ satisfy the recurrence relations

$$\begin{cases} d_{j\,,\,1} \;=\; 0 \text{ and } d_{1\,,\,l} \;=\; 1 & \text{for } j \geq 2 \text{ and } l \geq 1, \\ d_{j+1\,,\,l+1} = \dfrac{1}{l^2} d_{j\,,\,l} + d_{j+1\,,\,l} & \text{for } j,\, l \geq 1\,. \end{cases}$$

Using the fact that $d_{j-1\,,\,l} \;=\; 0$ for $l = 1, \cdots, j-2$ we get first for $j \geq 2$ the equality

$$d_{j\,,\,n_j} = \sum_{n_{j-1}=j-1}^{n_j - 1} \frac{1}{n_{j-1}^2} d_{j-1\,,\,n_{j-1}}$$

which we iterate to obtain

$$d_{k\,,\,l} = \sum_{n_{k-1}=k-1}^{l-1} \frac{1}{n_{k-1}^2} \sum_{n_{k-2}=k-2}^{n_{k-1}-1} \frac{1}{n_{k-2}^2} \cdots \sum_{n_2=2}^{n_3-1} \frac{1}{n_2^2} \sum_{n_1=1}^{n_2-1} \frac{1}{n_1^2}\,.$$

This leads to

$$\lim_{l\to\infty} d_{k\,,\,l} = \sum_{n_{k-1}>n_{k-2}>\cdots>n_2>n_1=1} \prod_{j=1}^{k-1} \frac{1}{n_j^2}$$

and we recognize in the right-hand side the number $\zeta(\{2\}_{(k-1)})$ whose value is equal to the right-hand side of (9) [1]. $\square$

Proof of Lemma 5
It suffices to prove that the numbers $e_{m\,,\,l}$ defined by

$$(-1)^{m+1} B_{2m}(\frac{1}{2} + \frac{1}{\pi}\text{Arcsin}\, x) = \sum_{l=0}^{\infty} e_{m\,,\,l}\, x^{2l} \tag{10}$$

satisfy $e_{m,\,l} > 0$ for all $m, l \in \mathbb{N}^*$. Using Taylor's formula and the evenness of function $B_{2m}(\frac{1}{2} + \frac{t}{\pi})$ we have

$$B_{2m}(\frac{1}{2} + \frac{t}{\pi}) = \sum_{k=0}^{m} \frac{1}{(2k)!}\, B_{2m}^{(2k)}(\frac{1}{2})\,(\frac{t}{\pi})^{2k} = \sum_{k=0}^{m} \binom{2m}{2k}\, B_{2m-2k}(\frac{1}{2})\,(\frac{t}{\pi})^{2k}$$

and the Taylor expansion of $(\text{Arcsin}\, x)^{2k}$ given in Sublemma 1 leads to

$$\begin{aligned} B_{2m}(\frac{1}{2} + \frac{\text{Arcsin}\, x}{\pi}) &= \sum_{k=0}^{m} \left(\binom{2m}{2k}\, B_{2m-2k}(\frac{1}{2})\,\pi^{-2k} \sum_{l=0}^{\infty} \frac{(2k)!}{(2l)!} 2^{2l-2k} b_{k\,,\,l} x^{2l} \right) \\ &= \frac{(2m)!}{(2\pi)^{2m}} \sum_{k=0}^{m} \left(\frac{(2\pi)^{2m-2k}}{(2m-2k)!} B_{2m-2k}(\frac{1}{2}) \sum_{l=0}^{\infty} \frac{2^{2l}}{(2l)!} b_{k\,,\,l} x^{2l} \right) . \end{aligned}$$

We then change the order of summation to get

$$(-1)^{m+1} B_{2m}(\frac{1}{2} + \frac{\text{Arcsin}\, x}{\pi}) = \frac{(2m)!}{(2\pi)^{2m}} \sum_{l=0}^{\infty} \frac{2^{2l}}{(2l)!} f_{m\,,\,l} x^{2l} \tag{11}$$

where

$$f_{m\,,\,l} = (-1)^{m+1} \sum_{k=0}^{m} \frac{(2\pi)^{2m-2k}}{(2m-2k)!} B_{2m-2k}(\frac{1}{2})\, b_{k\,,\,l}\,.$$

We prove by recurrence over m that $f_{m\,,\,l} > 0$ for $m,\, l \geq 1$. To this end we set $g_{m\,,\,l} = \dfrac{f_{m\,,\,l}}{((l-1)!)^2}$ for $m,\, l \geq 1$ and since $b_{0\,,\,l} = 0$ for $l \geq 1$ we have

$$\begin{aligned} g_{m+1\,,\,l+1} &= \frac{(-1)^{m+2}}{(l!)^2} \sum_{k=1}^{m+1} \frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!} B_{2m+2-2k}(\frac{1}{2})\, b_{k\,,\,l+1} \\ &= \frac{(-1)^{m+2}}{(l!)^2} \sum_{k=1}^{m+1} \frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!} B_{2m+2-2k}(\frac{1}{2}) \left(b_{k-1\,,\,l} + l^2\, b_{k\,,\,l} \right) \\ &= \frac{(-1)^{m+2}}{(l!)^2} \sum_{k=1}^{m+1} \frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!} B_{2m+2-2k}(\frac{1}{2})\, b_{k-1\,,\,l} + g_{m+1\,,\,l} \\ &= -\frac{(-1)^{m+1}}{(l!)^2} \sum_{k=0}^{m} \frac{(2\pi)^{2m-2k}}{(2m-2k)!} B_{2m-2k}(\frac{1}{2})\, b_{k\,,\,l} + g_{m+1\,,\,l} \\ &= -\frac{1}{l^2} g_{m\,,\,l} + g_{m+1\,,\,l} \end{aligned}$$

and this implies that

$$g_{m+1\,,\,l+1} + \frac{1}{l^2} g_{m\,,\,l} = g_{m+1\,,\,l} \quad \text{for } l \geq 1.$$

We have $g_{1,l} = f_{1,l} = 1$ for all $l \geq 1$. Let us suppose that $g_{m,l} > 0$ for all $l \geq 1$. Then $g_{m+1,l+1} < g_{m+1,l}$ and it follows that $g_{m+1,l} > \lim_{l\to\infty} g_{m+1,l}$. Thanks to Sublemma 2 we have

$$\begin{aligned}\lim_{l\to\infty} g_{m+1,l} &= (-1)^{m+2}\sum_{k=1}^{m+1}\frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!}B_{2m+2-2k}(\frac{1}{2})\frac{\pi^{2k-2}}{(2k-1)!}\\ &= (-1)^{m+2}\pi^{2m}\sum_{k=1}^{m+1}\frac{2^{2m+2-2k}}{(2m+2-2k)!(2k-1)!}B_{2m+2-2k}(\frac{1}{2}) \qquad (12)\end{aligned}$$

and using $B_j(\frac{1}{2}) = 0$ for all odd j and the formula

$$B_n(x+y) = \sum_{j=0}^{n}\binom{n}{j}B_j(x)y^{n-j}$$

we check that the sum which appears in (12) is equal to

$$\sum_{j=0}^{2m+1}\frac{2^j}{j!(2m+1-j)!}B_j(\frac{1}{2}) = \frac{2^{2m+1}}{(2m+1)!}\sum_{j=0}^{2m+1}\binom{2m+1}{j}B_j(\frac{1}{2})\left(\frac{1}{2}\right)^{2m+1-j} = \frac{2^{2m+1}}{(2m+1)!}B_{2m+1}(1) = 0\,.$$

Hence $g_{m,l} > 0$ for $m, l \geq 1$ and this implies, thanks to (11), that the numbers $e_{m,l}$ defined by (10) are positive for $m, l \geq 1$. $\square$

Properties of functions $\Psi^*_{n,2m-1}$

In order to apply the identity (2) we will need to indicate the dependence of functions $\Psi^*_{n,2m-1}$ defined in Lemma 4 with respect to x_i. We consider in detail the case $1 \leqslant n \leqslant l-1$ and use the notation $\Psi^*_{n,2m-1}(\{x_1,\cdots,x_n,x_l\},x)$. The case $n = l$ is treated similarly.

Lemma 6
Let $0 < c < 1$, $1 \leqslant n \leqslant l-1$ and $\Psi^*_{n,2m-1}$ be the function defined in Lemma 4. Then

a) $(-1)^{n+m}\Psi^*_{n,2m-1}(\{x_1,\cdots,x_n,x_l\},x_l) > 0$ for $0 < x_1 < \cdots < x_n < x_l$.

b) $(-1)^{n+m}\frac{\partial}{\partial x_k}\Psi^*_{n,2m-1}(\{x_1,\cdots,x_n,x_l\},x_l) > 0$ for $k = 1,2,\cdots,n$.

c) $(-1)^{n+m}\frac{d}{ds}\Psi^*_{n,2m-1}(\{s,\pi+s,\cdots,(n-1)\pi+s,(l-1)\pi+s\},(l-1)\pi+s) > 0$.

d) For $0 = x_0 < x_1 < \cdots < x_l < \frac{n\pi}{c}$ we have the identity[4]

$$\sum_{k=1}^{\infty}(-1)^{n+k}\Psi^*_{n,2k-1}(\{x_1,\cdots,x_n,x_l\},x_l)c^{2k-1} = (-1)^{n+1}\sum_{j=0}^{n}\mu^*_{n,j}\frac{\cos(c\,x_j)}{\sin(c\,x_l)}. \qquad (13)$$

e) For $n > c(l-1)$ there exists a unique $s \in]0,(\frac{n}{c}-(l-1))\pi[$ such that

$$\sum_{k=1}^{\infty}(-1)^{n+k}\Psi^*_{n,2k-1}(\{s,\pi+s,\cdots,(n-1)\pi+s,(l-1)\pi+s\},(l-1)\pi+s)c^{2k-1} = 1\,. \qquad (14)$$

[4]It follows from the proof that the singularities of the right-hand side when $c\,x_l = j\,\pi$ where $j = 1,\cdots,n-1$ are removable.

Proof

a) By definition of $\theta_{k,2m-1}$ we have

$$\theta_{k,2m-1}(x_l) = \frac{2^{2m} x_l^{2m-1}}{(2m)!}(B_{2m}(1) - B_{2m}(\frac{1}{2} + \frac{x_k}{2x_l}))$$

and then

$$(-1)^{n+m}\,\Psi^*_{n,2m-1}(\{x_1, \cdots, x_l\}, x_l) = \frac{2^{2m} x_l^{2m-1}}{(2m)!}\sum_{k=0}^{n}(-1)^{n+m}\mu^*_{n,k}\,\Big(B_{2m}(1) - B_{2m}(\frac{1}{2} + \frac{x_k}{2x_l})\Big)$$

$$= \frac{2^{2m} x_l^{2m-1}}{(2m)!}\left(\prod_{1\leqslant j\leqslant n}\sin^2(\pi\frac{x_j}{2x_l})\right)\sum_{k=0}^{n}\frac{(-1)^m B_{2m}(1) + (-1)^{m+1}B_{2m}(\frac{1}{2} + \frac{x_k}{2x_l})}{\prod\limits_{\substack{0\leqslant j\leqslant n\\ j\neq k}}\Big(\sin^2(\pi\frac{x_k}{2x_l}) - \sin^2(\pi\frac{x_j}{2x_l})\Big)}.$$

The first two terms of the right-hand side are positive and setting $t_k = \sin^2(\pi\frac{x_k}{2x_l})$ for $k = 0, 1, \cdots, n$ the third term which can be rewritten as

$$\sum_{k=0}^{n}\frac{(-1)^m B_{2m}(1) + (-1)^{m+1}B_{2m}(\frac{1}{2} + \frac{1}{\pi}\mathrm{Arcsin}\sqrt{t_k}\,)}{\prod\limits_{\substack{0\leqslant j\leqslant n\\ j\neq k}}(t_k - t_j)}$$

is positive thanks to Lemmas 3 and 5.

b) This is a consequence of the rewriting of the function $(-1)^{n+m}\,\Psi^*_{n,2m-1}$ given in a) and the relations $\frac{dt_k}{dx_k} > 0$ for $k = 1, \cdots, n$.

c) We introduce the functions $x^*_k(s) \overset{def}{=} (k-1)\pi + s$ for $k = 1, \cdots, l$ and we use the above arguments and the relations $\frac{d}{ds}\left(\frac{x^*_k(s)}{x^*_l(s)}\right) > 0$ for $k = 1, \cdots, n$.

d) Since $c\,x_l < n\pi$ and $\Psi^*_{n,2k-1} = O_l((\frac{x_l}{n\pi})^{2k-1})$ the left-hand side of (13) is well defined. By the proof of Lemma 4 we have

$$\Psi^*_{n,2k-1}(\{x_1, \cdots, x_n, x_l\}, x_l) = (-1)^{k+1}\,4\sum_{p=n}^{\infty}\sum_{j=0}^{n}\mu^*_{n,j}\,a_{p\,,\,j}(-1)^p\,\frac{1}{x_l}\left(\frac{x_l}{p\pi}\right)^{2k}$$

where $a_{p\,,\,j} = \sin^2(p\pi\frac{x_j}{2x_l}) - \frac{(-1)^{p+1}+1}{2}$. It is sufficient to prove (13) for x_l such that $c\,x_l \neq j\pi$ for $j = 1, \cdots, n-1$. For such an x_l it results that

$$\sum_{k=1}^{\infty}(-1)^{n+k}\Psi^*_{n,2k-1}(\{x_1, \cdots, x_n, x_l\}, x_l)c^{2k-1} = (-1)^{n+1}\,4\sum_{p=n}^{\infty}\sum_{j=0}^{n}\mu^*_{n,j}\,a_{p\,,\,j}\,(-1)^p\,\frac{1}{cx_l}\sum_{k=1}^{\infty}\left(\frac{cx_l}{p\pi}\right)^{2k}$$

$$= (-1)^{n+1}\,4\sum_{p=n}^{\infty}\sum_{j=0}^{n}\mu^*_{n,j}\,a_{p\,,\,j}\,(-1)^p\,\frac{cx_l}{p^2\pi^2 - c^2x_l^2}$$

$$= (-1)^{n+1}\,4\sum_{p=1}^{\infty}\sum_{j=0}^{n}\mu^*_{n,j}\,a_{p\,,\,j}\,(-1)^p\,\frac{cx_l}{p^2\pi^2 - c^2x_l^2}$$

$$= (-1)^{n+1}\,2\sum_{p=1}^{\infty}\sum_{j=0}^{n}\mu^*_{n,j}\,\cos(p\pi\frac{x_j}{x_l})(-1)^{p-1}\,\frac{cx_l}{p^2\pi^2 - c^2x_l^2}$$

where we use successively the absolute convergence to change the order of summation and the relations $\sum_{j=0}^{n} \mu_{n,j}^* a_{p\,,j} = 0$ for $p = 1, \cdots, n-1$, $\sin^2(\frac{\alpha}{2}) = \frac{1}{2} - \frac{1}{2}\cos(\alpha)$ and $\sum_{j=0}^{n} \mu_{n,j}^* = 0$.
We finally make use of the identity ([2], formula (17.3.10))

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2a^2 - b^2}\cos(nx) = \frac{\pi}{2ab}\frac{\cos(\frac{b}{a}x)}{\sin(\pi\frac{b}{a})} - \frac{1}{2b^2} \quad \text{where} \quad \frac{b}{a} \notin \mathbb{Z} \quad \text{and} \quad -\pi \leqslant x \leqslant \pi$$

to verify that the last term of the above equalities is equal to the right-hand side of (13).

e) Let $c(l-1) < n \leqslant l$, $\widetilde{s} = (\frac{n}{c} - (l-1))\pi$ and h be the function defined for $s \in]0, \widetilde{s}\,[$ by the left-hand side of equation (14). From c) the function h is strictly increasing on $]0, \widetilde{s}\,[$. Otherwise, when s tends to 0_+, the coefficient $\mu_{n,j}^*$ tends to -1 if $j=1$ and to 0 if $j = 2, \cdots, n$. We deduce easily that $\lim_{s \to 0_+} h(s) = 0$. Moreover, setting $\tau_k = \sin^2(\pi\frac{x_k^*(\widetilde{s})}{2x_l^*(\widetilde{s})})$ and denoting by T_n the Chebyshev polynomial of degree n, we check that

$$\begin{aligned}
\lim_{s \to \widetilde{s}_-} (-1)^{n+1}\,\sin(c\,x_l^*(s))\,h(s) &= (-1)^n \left(\prod_{1 \leqslant j \leqslant n} \tau_j\right) \sum_{k=0}^{n} \frac{\cos(2n\mathrm{Arcsin}\sqrt{\tau_k}\,)}{\prod_{\substack{0 \leqslant j \leqslant n \\ j \neq k}} (\tau_k - \tau_j)} \\
&= (-1)^n \left(\prod_{1 \leqslant j \leqslant n} \tau_j\right) \sum_{k=0}^{n} \frac{T_n(1-2\tau_k)}{\prod_{\substack{0 \leqslant j \leqslant n \\ j \neq k}} (\tau_k - \tau_j)} = 2^{2n-1} \prod_{1 \leqslant j \leqslant n} \tau_j > 0
\end{aligned}$$

where the last equality is a consequence of Lemma 3. It follows that $\lim_{s \to \widetilde{s}_-} h(s) = \infty$ and hence there exists a unique $s \in]0, \widetilde{s}\,[$ such that $h(s) = 1$. □

Proof of Theorem

We assume $s \in]0, s^*(c,l)[$, which will lead to a contradiction. Let n such that $s^*(c,l) = s^{(n)}(c,l)$ where we assume $n \leqslant l-1$. The case $n = l$ is treated similarly.
Using definition of $\mu_{n,k}^*$ given in Lemma 4 and relation (13) one checks that the equation $f_c^{(l,n)}(s) = 0$ is equivalent to

$$\sum_{k=1}^{\infty} (-1)^{n+k}\Psi_{2k-1}^*(\{x_1^*(s), \cdots, x_{l-1}^*(s), x_l^*(s)\}, x_l^*(s))c^{2k-1} = 1\,.$$

Letting m go to infinity in (2) and taking into account Lemma 2 d) and Lemma 6 we have successively

$$\begin{aligned}
f(0) &\leqslant \sum_{k=1}^{\infty} \left|\Psi_{n,2k-1}^*(\{x_1, \cdots, x_n, x_l^*(s)\}, x_l^*(s))\right| c^{2k-1} \\
&= \sum_{k=1}^{\infty} (-1)^{n+k}\Psi_{n,2k-1}^*(\{x_1, \cdots, x_n, x_l^*(s)\}, x_l^*(s))c^{2k-1} \\
&\leqslant \sum_{k=1}^{\infty} (-1)^{n+k}\Psi_{n,2k-1}^*(\{x_1^*(s), \cdots, x_n^*(s), x_l^*(s)\}, x_l^*(s))c^{2k-1} \\
&< \sum_{k=1}^{\infty} (-1)^{n+k}\Psi_{n,2k-1}^*(\{x_1^*(s^*(c,l)), \cdots, x_n^*(s^*(c,l)), x_l^*(s^*(c,l))\}, x_l^*(s^*(c,l)))c^{2k-1} = 1\,.
\end{aligned}$$

This is a contradiction since $f(0) = 1$. □

Conclusion

In this paper we state our results for even functions. Lemmas 1, 2 and 4 extend to arbitrary functions and the analogue of Lemma 6 should lead to the generalization of our theorem.
In view of applications, it is important to impose, for instance, conditions only on the $2l$ first derivatives. Our method applies to this case, an additional term simply appears.

Acknowledgements

I would like to thank Professor Jacques Rappaz for accepting to publish this work and Professor Jean Descloux and Socrates Litsios for their valuable comments.

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