Data-driven model reference control with asymptotically guaranteed stability

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SUMMARY

This paper presents a data-driven controller tuning method that includes a set of constraints for ensuring closed-loop stability. The approach requires a single experiment and can also be applied to nonminimum-phase and unstable systems. The tuning scheme generates an estimate of the closed-loop output error that is used to minimize an approximation of the model reference control problem. The correlation approach is used to deal with the influence of measurement noise. For linearly parameterized controllers, this leads to a convex optimization problem. A sufficient condition for closed-loop stability is introduced, which can be included in the optimization problem for control design. As the data length tends to infinity, closed-loop stability is guaranteed. The quality of the estimated controller is analyzed for finite data length. The effectiveness of the proposed method is demonstrated in simulation as well as experimentally on a laboratory-scale mechanical setup.

KEY WORDS: Data-driven controller tuning; Model reference control; Stability; Correlation-based tuning; Convex optimization

1. INTRODUCTION

Consider the control problem with the performance specifications given in terms of a reference model. The objective is to design a controller such that the closed-loop system resembles the reference model. The standard model-based solution to this problem requires identification of a plant model, which is then used to compute the controller that minimizes the error between the closed-loop system and the reference model. This approach thus uses two optimizations, one in the identification step and a second one in the controller design. Furthermore, a controller-order reduction step might be needed before implementation.

In recent years, several data-driven techniques have been proposed as an alternative to these model-based approaches [1, 2, 3, 4]. In a data-driven approach, the aforementioned steps of

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controller design are lumped together, resulting in a direct “data-to-controller” algorithm that uses a single optimization. Compared to a model-based approach, the optimization in the plant identification step is omitted and the problem of undermodeling of the plant is avoided. Furthermore, the designed controller does not depend on the structure of the model. The order of the resulting controller can be fixed, in contrast to some model-based methods, where the controller order is related to the model order.

A data-driven approach to the model reference problem leads in general to a non-convex optimization problem. In Iterative Feedback Tuning (IFT) [1] and Iterative Correlation-based Tuning (ICbT) [3], a gradient approach is used to find a (local) optimum of the control objective. At each iteration, an experiment is used to evaluate the criterion or estimate the gradient, thus leading to an iterative scheme. The unfalsified control concept [5] allows evaluation of a closed-loop error with a single experiment. However, this error corresponds to a virtual reference signal, whose spectrum depends on the controller and the unknown plant. Consequently, this approach cannot be used to minimize a general 2-norm control criterion.

The concept of virtual reference controller design as introduced in [6] can be used to minimize an approximation of the model reference criterion, for which the global minimum can be found using a non-iterative scheme. An extension to this original method with an appropriate weighting for fixed-order controllers is named Virtual Reference Feedback Tuning (VRFT) [4]. The method is developed for noise-free measurements. When the data is corrupted with noise, this scheme leads to a specific identification problem, for which the well-known prediction-error methods are not consistent [7]. For noisy measurements, the use of a second experiment has been proposed, which is statistically inefficient. Note that VRFT has also been extended to 2 degree-of-freedom controllers [8] and to nonlinear plants [9]. This paper presents a non-iterative controller-tuning scheme that uses the correlation approach to deal with the measurement noise.

One of the main difficulties of data-driven approaches regards the stability of the closed-loop plant. In model-based approaches, the model of the plant can be used to analyze whether the controller is suitable, before actual implementation. In a data-driven method, since no model is available, stability is typically not guaranteed before implementation of the controller. Several a posteriori tests to verify closed-loop stability have been proposed, e.g. [10, 11, 12], where stability is verified after controller computation and before actual implementation. This paper presents the first known attempt to incorporate a stability condition in data-driven controller design. The stability condition is implemented as a set of convex constraints that can be added to any data-driven controller tuning scheme for linearly parametrized controllers. In this paper the stability condition is added to a non-iterative scheme based on the correlation approach, labeled Correlation-based Tuning with Guaranteed Stability, CbT-GS.

The stability condition is formulated in terms of the H_∞-norm of a particular error function. The constraint is implemented using the discrete Fourier transform (DFT) of auto- and cross-correlation functions. This leads to a convex optimization problem whose solution is consistent. Furthermore, the computational load with this DFT estimate is small, compared to methods that use Toeplitz matrices to estimate the H_∞-norm [13, 14, 2]. Consequently, problems with much larger data size can be handled [15].

The proposed data-driven controller tuning method is applicable to stable, unstable as well as nonminimum-phase plants. It guarantees a stabilizing solution as the number of data tends to infinity. In practice, only a finite number of data can be used, and the results depend strongly on the quality of the estimates, which is determined by the excitation signal used in the experiment. The method is developed for non-periodic signals and the implementation for periodic data is summarized. Indications on how to choose the design parameters follow from analysis of the estimates based on a finite number of data.

The paper is organized as follows. In Section 2, a constrained approximate model reference problem is presented, that guarantees a stabilizing solution. Section 3 describes a data-driven solution to this problem. For stable systems, an open-loop tuning scheme is presented; for unstable or nonminimum-phase systems, a closed-loop scheme is proposed. In Section 4, implementation using the correlation approach is developed. The simulation example in Section 5 shows the performance of CbT-GS. Using the same example, the effectiveness of adding the stability constraints is demonstrated with regard to VRFT. Section 5 also illustrates the application of CbT-GS to a laboratory-scale torsional plant. Conclusions are provided in Section 6.

2. MODEL REFERENCE CONTROL WITH GUARANTEED STABILITY

2.1. Model reference control problem

Consider the unknown linear SISO plant $G(q^{-1})$, where q^{-1} denotes the backward shift operator. Specifications for the controlled plant are given as a stable strictly proper reference model $M(q^{-1})$. The objective is to design a linear, fixed-order controller $K(q^{-1}, \rho)$, with parameters ρ, for which the controlled plant resembles the reference model $M(q^{-1})$.

This can be achieved by minimizing the two-norm of the difference between the reference model and the achieved closed-loop system:

$$J_{mr}(\rho) = \left\| F\left[M - \frac{K(\rho)G}{1 + K(\rho)G}\right] \right\|_2^2 \tag{1}$$

with F a user defined weighting filter. Note that the objective is to design a fixed-order controller and that $J_{mr}(\rho) = 0$ can in general not be achieved.

The model reference criterion (1) is non-convex with respect to the controller parameters ρ. An approximation that is convex for linearly parameterized controllers can be defined using the reference model M as illustrated next. M can be represented as:

$$M = \frac{K^*G}{1 + K^*G}. \tag{2}$$

The backward shift operator is omitted here and will be omitted in the sequel. K^* is the ideal controller, which is defined indirectly by G and M:

$$K^* = \frac{M}{G(1 - M)}. \tag{3}$$

This controller K^* exists since M is strictly proper, i.e. $M \neq 1$. K^* might be of very high order since it depends on the unknown and possibly high-order plant G. Furthermore, it might not stabilize the plant internally and it might be non-causal. Note, however, that the unknown ideal controller will only be used for analysis and the results will be valid also for a non-causal K^*. Furthermore, since M is strictly proper, $K^*G = M(1 - M)^{-1}$ is causal. The ideal sensitivity function is then given by

$$\frac{1}{1 + K^*G} = 1 - M. \tag{4}$$

Using (2), the model reference criterion (1) can be expressed as:

$$J_{mr}(\rho) = \left\| F \left[\frac{K^*G - K(\rho)G}{(1+K^*G)(1+K(\rho)G)} \right] \right\|_2^2 \tag{5}$$

Approximation of $\frac{1}{1+K(\rho)G}$ by the ideal sensitivity function (4) leads to the following approximation of the model reference criterion:

$$J(\rho) = \left\| F \left[\frac{K^*G - K(\rho)G}{(1+K^*G)^2} \right] \right\|_2^2 = \left\| F(1-M)[M - K(\rho)(1-M)G] \right\|_2^2. \tag{6}$$

For a linearly parameterized $K(\rho)$, this approximation is convex with respect to ρ. The approximation is good if the difference between $K(\rho)$ and the ideal controller K^* can be made small. This approximation has been used in model reduction and controller reduction, see [16] for an overview. A similar approximation in the H_∞ framework is for example used in [17], an H_2 example can be found in [18]. The approximation has also been used in data-driven controller tuning [4]. The quality of the approximation is discussed in [4].

The controller that minimizes $J(\rho)$ is denoted by $K(\rho_0)$ and will be referred to as the optimal controller. Note that, if the ideal controller K^* is in the set of controllers given by $K(\rho)$, the optimal $K(\rho_0)$ is given by $K(\rho^*) = K^*$, i.e. $\rho_0 = \rho^*$. In this case, the frequency weighting does not affect the result since $K(\rho^*)G(1-M) = M$ and therefore both $J(\rho^*) = 0$ and $J_{mr}(\rho^*) = 0$; the approximate model reference criterion $J(\rho)$ and $J_{mr}(\rho)$ have the same optimum, ρ^*.

2.2. Stability constraint

There is no guarantee that a controller determined by minimizing $J(\rho)$ actually stabilizes the plant. Instability can occur if the reference model is chosen inappropriately or if the measurements are strongly affected by noise. The ideal controller K^* is defined indirectly from G and M as shown in (3). Whether K^* stabilizes the plant depends on both the plant G and the choice of reference model M. If the plant is nonminimum phase, internal stability can only be guaranteed when M contains the unstable zeros of G. This clearly makes the choice of an appropriate M difficult in a data-driven approach.

Even if the ideal controller K^* stabilizes the plant, this is not necessarily the case for the optimal controller $K(\rho_0)$ (see [19] for an example where K^* was not in the controller set). Furthermore, if the optimal controller $K(\rho_0)$ stabilizes the plant, an estimate of $K(\rho_0)$ based on noisy data might not be stabilizing. In the following, a sufficient condition that guarantees stability of the resulting closed-loop system is proposed. It will be shown in Section 4 that an estimate of this condition leads to a set of convex constraints that can be added to any data-driven controller tuning scheme for linearly parametrized controllers.

Consider a stabilizing controller K_s. The closed-loop plant for this controller is given by:

$$M_s = \frac{K_s G}{1 + K_s G}. \tag{7}$$

The closed-loop system with controller $K(\rho)$ can be represented as illustrated in Fig. 1. In the following, stability is defined as having all poles within the open unit circle. Define

$$\begin{aligned} \Delta(\rho) :=& M_s - K(\rho)G\,(1 - M_s) \\ \delta(\rho) :=& \|\Delta(\rho)\|_\infty. \end{aligned} \tag{8}$$

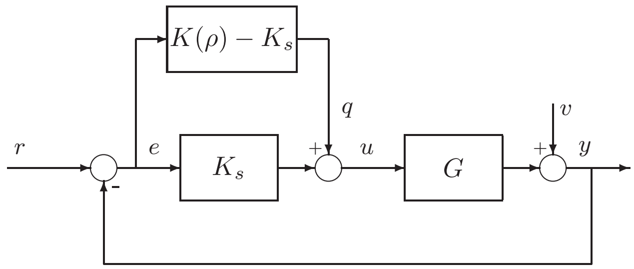

Figure 1. Closed-loop system with controller $K(\rho)$ and explicit representation of the controller error $K(\rho) - K_s$

Theorem 1. *The controller $K(\rho)$ stabilizes the plant G if*

1. $\Delta(\rho)$ *is stable*
2. $\exists \delta_N \in \,]0, 1[$ *such that* $\delta(\rho) \leqslant \delta_N$

Proof: If Condition 1 is satisfied, all transfer functions of the loop opened at q are stable, since K_s stabilizes the plant, i.e. the transfer functions from $r(t), v(t)$ and $q(t)$ to $e(t), y(t), u(t)$ and $q(t)$ are stable (see Fig. 1.). The sufficient condition for stability of the closed-loop interconnection follows from the small-gain theorem [20] : the interconnection is stable if

$$\left\| \frac{-(K(\rho) - K_s)G}{1 + K_s G} \right\|_\infty < 1. \tag{9}$$

This is the H_∞-norm of the transfer function from q back to q. Replacing $\frac{K_s G}{1+K_s G}$ by M_s and $\frac{1}{1+K_s G}$ by $1 - M_s$ gives

$$\left\| \frac{-(K(\rho) - K_s)G}{1 + K_s G} \right\|_\infty = \delta(\rho).$$

■

Theorem 1 thus follows from the small-gain theorem. Similar conditions for stability have been used for controller reduction (see for example [21], p. 491). If Condition 1 is satisfied, Condition 2 is sufficient for closed-loop stability of the feedback system of $K(\rho)$ and G. Condition 1 can easily be enforced for different controllers. A few examples are given next:

- If $K(\rho)$ is stable, Condition 1 is satisfied since K_s stabilizes the plant. Consequently $\Delta(\rho)$ is stable.
- If $K(\rho)$ contains an integrator, Condition 1 is satisfied if $G(1 - M_s)$ contains a zero at 1. This is for example satisfied if K_s contains an integrator.

If Condition 1 is satisfied, the sufficient Condition 2 can be used to guarantee a stabilizing solution to the model reference problem of Section 2.1. This leads to the following optimization problem:

$$\begin{aligned} \rho_s &= \arg \min_{\rho \in \mathcal{D}_K} J(\rho) \\ &\text{subject to } \delta(\rho) \leqslant \delta_N \end{aligned} \tag{10}$$

The optimal solution ρ_s defines a stabilizing controller and will be referred to as the stabilizing optimum. In the following, a data-driven approach for solving (10) is presented.

Remark: In practice, only an estimate of $\delta(\rho)$ will be available. Regardless of whether this estimate is based on a model of the plant G or estimated directly from data as proposed in this paper, it will be uncertain. In order to guarantee stability in practice, the estimation errors will have to be taken into account through the choice of δ_N.

3. DATA-DRIVEN CONTROLLER TUNING SCHEME

Let the controller be linearly parametrized

$$K(q^{-1}, \rho) = \beta^T(q^{-1})\rho, \rho \in \mathcal{D}_K \tag{11}$$

where the set $\mathcal{D}_K$ is compact and $\beta(q^{-1})$ is a vector of stable linear discrete-time transfer operators:

$$\beta(q^{-1}) = [\beta_1(q^{-1}), \beta_2(q^{-1}), \ldots, \beta_{n_\rho}(q^{-1})]^T. \tag{12}$$

With this structure of $K(\rho)$, the approximate model reference criterion $J(\rho)$ is convex in the controller parameters ρ. Using a data-driven controller tuning scheme, the global optimum of this criterion can be found using only one set of measured data. For stable minimum-phase systems, one open-loop experiment is sufficient (Section 3.1). For nonminimum-phase or unstable systems, one closed-loop experiment is needed (Section 3.2).

3.1. Tuning scheme for stable minimum-phase plants

Theorem 1 is based on the small-gain theorem and requires the closed-loop system M_s to be internally stable. For stable minimum-phase plants, any stable reference model M defines an ideal controller K^* (3) that internally stabilizes the system. The reference model can therefore be used to define sufficient conditions for stability.

Lemma 1. *Let M_s be given by M. The controller $K(\rho)$ stabilizes the stable minimum-phase plant G if $\Delta(\rho) = M_s - K(\rho)G(1 - M_s) = M - K(\rho)G(1 - M)$ is stable and $\exists \delta_N \in \ ]0, 1[$ such that*

$$\delta(\rho) = \|M_s - K(\rho)(1 - M_s)G\|_\infty = \|M - K(\rho)(1 - M)G\|_\infty \leqslant \delta_N \tag{13}$$

Proof: Follows from Theorem 1 upon replacing K_s by the stabilizing ideal controller K^*. ■

Remark: K^* given in (3) might be non-causal, but K^*G is always causal. The small-gain theorem requires causality because algebraic loops will occur for non-causal functions. However, since K^*G is always causal, no algebraic loop occurs in the interconnection of Fig. 1 and Theorem 1 remains valid.

Condition (13) leads to the following optimization problem:

$$\begin{gathered} \rho_s = \arg \min_{\rho \in \mathcal{D}_K} J(\rho) \\ \text{subject to} \\ \|M - K(\rho)(1 - M)G\|_\infty \leqslant \delta_N \end{gathered} \tag{14}$$

Remark: Condition (13) is sufficient but not necessary and therefore conservative. The optimal controller $K(\rho_0)$ that minimizes $J(\rho)$ might stabilize the system but not meet condition

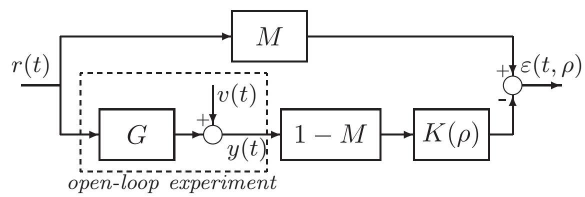

Figure 2. Tuning scheme for the model reference control problem using a single open-loop experiment

(13). However, this indicates that the distance between $K(\rho)$ and K^* cannot be made small. In this case, the approximate model reference criterion (6) is not a good approximation of (1).

If $K(\rho)$ is stable, $\Delta(\rho)$ is stable by definition and $K(\rho_s)$ is guaranteed to stabilize the plant. If the plant G or the controller $K(\rho)$ contains one or several integrators, the above scheme remains applicable provided the reference model is chosen with care. Let n_i be the number of integrators in the loop function KG. It is then easily verified that K^* stabilizes G, and $\Delta(\rho)$ is stable if $1 - M$ has $n_z \geqslant n_i$ zeros at 1. The reference model M needs to be chosen such that this condition is satisfied. Note that, if $n_i = 1$, all reference models with unity static gain satisfy this condition.

In a data-driven approach, the error $\varepsilon(t, \rho)$ given by the tuning scheme of Fig. 2 can be used to compute the optimal controller. $\varepsilon(t, \rho)$ can be expressed in terms of the exogenous signals $r(t)$ and $v(t)$ as follows:

$$\varepsilon(t, \rho) = Mr(t) - K(\rho)(1 - M)y(t) = \big[M - K(\rho)(1 - M)G\big]r(t) - K(\rho)(1 - M)v(t) \quad (15)$$

In the resulting parameter estimation problem, the input to the function to be identified, $K(\rho)$, is affected by noise, in contrast to classical identification problems where its output is affected by noise. For this particular identification problem, prediction-error methods are inconsistent. The correlation approach will be used to reduce the effect of noise on the estimated controller parameters. This approach is applicable to deterministic as well as stochastic reference signals, both non-periodic and periodic.

Note that the transfer function between $r(t)$ and $\varepsilon(t, \rho)$ is equal to the transfer function defining $\delta(\rho)$ in (13). Hence, the available signals $r(t)$ and $\varepsilon(t, \rho)$ can also be used to estimate $\delta(\rho)$. It will be shown that a spectral estimate leads to a set of convex constraints on the controller parameters ρ.

3.2. Tuning scheme for nonminimum-phase or unstable plants

For nonminimum-phase or unstable plants, an arbitrary reference model M does not define a stabilizing ideal controller K^*. For such plants, Lemma 1 is not applicable, and the optimization problem (10) needs to be used instead of (14). In (10), the control criterion $J(\rho)$ is defined using the (arbitrary) reference model M, whereas the constraint for stability uses M_s. If a stabilizing controller K_s is available, the closed-loop interconnection of G and K_s represents M_s given in (7). In order to estimate $\delta(\rho)$, a set of input-output data of the transfer function $M_s - K(\rho)(1 - M_s)G$ is sufficient.

Data from an experiment on the plant controlled by a stabilizing controller K_s is assumed available. Note that K_s and M_s might be unknown. Consider the tuning scheme shown in Fig.

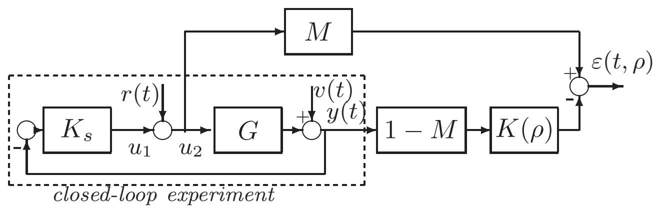

Figure 3. Tuning scheme for model reference control problem using one closed-loop experiment

3. The excitation signal is applied directly to the input of the plant. The data set consists of the exogenous excitation signal $r(t)$, the output of the controller $u_1(t)$, the resulting input to the plant $u_2(t) = u_1(t) + r(t)$, and the output of the controlled plant $y(t)$. The error $\varepsilon(t, \rho)$, which reads

$$\varepsilon(t, \rho) = M u_2(t) - K(\rho)(1 - M)y(t), \qquad (16)$$

can be used to compute the optimal controller. Again, prediction-error methods are inconsistent. As for the open-loop scheme, the correlation approach will be used to reduce the effect of noise on the estimated controller parameters.

A second error signal $\varepsilon_s(t, \rho)$, which will be used in the stability constraint, is defined as:

$$\varepsilon_s(t, \rho) = -u_1(t) - K(\rho)y(t) = \big(M_s - K(\rho)(1 - M_s)G\big)r(t) + (K_s - K(\rho))(1 - M_s)v(t) \qquad (17)$$

The transfer function between $r(t)$ and $\varepsilon_s(t, \rho)$ is equal to the transfer function defining $\delta(\rho)$ in (8). Hence, the signals available from the scheme of Fig. 3 can be used to estimate $\delta(\rho)$.

Remarks:

- In the case of stable minimum-phase plants, violation of condition (13) indicates that the model reference criterion was inappropriate. This is no longer the case for the closed-loop scheme of Fig. 3, where violation of Condition 2 of Theorem 1 simply implies that closed-loop stability cannot be guaranteed, because the distance between the controller $K(\rho)$ and the stabilizing controller K_s is not small. This result agrees with ideas from iterative identification and control, e.g. [22, 23]. In [23], the term “safe controller changes” is used to denote an acceptable controller change that ensures a stability margin. The idea is that, by limiting the change in the controller, one can also limit the degradation that can occur in the actual closed-loop system.
- A test that uses experimental closed-loop data to verify whether a controller stabilizes the plant is proposed in [10]. The method uses coprime factorization and can handle unstable systems as well as unstable controllers. In the specific case of a stable controller, the experiment proposed in [10] corresponds to the scheme of Fig. 3. The transfer function considered in our stability criterion is the same as the transfer function considered in the stability test in [10]. However, the stability tests are different. In [10], both phase and amplitude are taken into account. The Nyquist stability criterion then leads to a non-conservative test, which corresponds to verifying whether $M_s - K(\rho)(1 - M_s)G$ does not encircle the point -1 in the complex plane. A frequency-domain model of $M_s - K(\rho)(1 - M_s)G$ is identified and used for verification. In this work, the stability

criterion uses the small-gain theorem, which leads to a conservative result. However, the resulting H_∞-norm constraint is convex and can be added to a convex controller optimization. The non-conservative test using both amplitude and phase information would lead to non-convex constraints.

4. IMPLEMENTATION USING THE CORRELATION APPROACH

Implementation of the tuning scheme for stable minimum-phase systems is discussed first. The case of unstable or minimum-phase systems is similar and will be addressed briefly in Section 4.2.

The ideal controller $K(\rho^*)$ achieves $M = K(\rho^*)G(1-M)$. As a result, the error signal (15) becomes filtered noise:

$$\varepsilon(t,\rho^*) = -K(\rho^*)(1-M)v(t) \tag{18}$$

Since $v(t)$ is not correlated with the reference $r(t)$, the ideal error $\varepsilon(t,\rho^*)$ will not be correlated with $r(t)$ either. Hence, the objective is to tune the controller parameters ρ such that $\varepsilon(t,\rho)$ and $r(t)$ become uncorrelated.

4.1. Implementation for stable minimum-phase plants

Let the plant G be excited by $r(t)$ as illustrated in Fig. 2. The output of the plant is affected by noise, $y(t) = Gr(t) + v(t)$. The signals $r(t)$ and $y(t)$ of length N are available. We assume the following:

A1 The reference signal is quasi-stationary, i.e.

$$R_r(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} r(t-\tau)r(t)$$

exists for all τ

A2 The spectrum of the reference signal $r(t)$ satisfies $\Phi_r(\omega) > 0, \forall\omega$, where

$$\Phi_r(\omega) = \sum_{\tau=-\infty}^{\infty} R_r(\tau)e^{-j\tau\omega}$$

and the infinite sum exists.

A3 The noise $v(t)$ can be represented as $v(t) = H(q^{-1})e(t)$, where $e(t)$ is a zero-mean white noise signal with variance σ^2 and bounded fourth moments. $H(q^{-1})$ is stable.

A4 The noise is not correlated with the input, i.e.

$$R_{rv}(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} E\left\{r(t-\tau)v(t)\right\} = 0 \tag{19}$$

for all τ.

Assumption **A1** includes deterministic as well as stochastic signals, i.e. $r(t)$ can be a realization of a stochastic process. $r(t)$ is non-periodic under assumption **A2**.

The error $\varepsilon(t,\rho)$ is calculated according to the tuning scheme of Fig. 2 and is given by (15). The vector of instrumental variables $\zeta(t)$, correlated with $r(t)$ and uncorrelated with $v(t)$, is defined as:

$$\zeta(t) = [r_W(t+l_1), r_W(t+l_1-1), \ldots r_W(t), r_W(t-1), \ldots, r_W(t-l_1)]^T \tag{20}$$

where l_1 is a sufficiently large integer and $r_W(t)$ is the filtered reference signal $r_W(t) = W(q^{-1})r(t)$. The correlation function is defined as

$$f_{N,l_1}(\rho) = \frac{1}{N}\sum_{t=1}^{N} \zeta(t)\varepsilon(t,\rho) \tag{21}$$

and the correlation criterion $J_{N,l_1}(\rho)$ as

$$J_{N,l_1}(\rho) = f_{N,l_1}^T(\rho) f_{N,l_1}(\rho). \tag{22}$$

The H_∞-norm $\delta(\rho)$ can be estimated using spectral estimates. The power spectrum of the reference signal $r(t)$ can be estimated for $\omega_k = 2\pi k/(2l_2+1)$, where $k = 0, \ldots, l_2+1$:

$$\hat{\Phi}_r(\omega_k) = \sum_{\tau=-l_2}^{l_2} \hat{R}_r(\tau)e^{-j\tau\omega_k},$$

and $\hat{R}_r(\tau)$ is an estimate of the auto-correlation $R_r(\tau)$ of $r(t)$:

$$\hat{R}_r(\tau) = \frac{1}{N}\sum_{t=1}^{N} r(t-\tau)r(t), \text{ for } \tau = -l_2, \ldots, l_2, \tag{23}$$

where l_2 defines the length of the rectangular window. The cross-spectrum between $r(t)$ and $\varepsilon(t,\rho)$ can be estimated as

$$\hat{\Phi}_{r\varepsilon}(\omega_k,\rho) = \sum_{\tau=-l_2}^{l_2} \hat{R}_{r\varepsilon}(\tau,\rho)e^{-j\tau\omega_k},$$

using an estimate of the cross-correlation $R_{r\varepsilon}(\tau,\rho)$:

$$\hat{R}_{r\varepsilon}(\tau,\rho) = \frac{1}{N}\sum_{t=1}^{N} r(t-\tau)\varepsilon(t,\rho), \quad \tau = -l_2, \ldots, l_2.$$

An estimate of $\delta(\rho)$ based on these cross-spectra is given by:

$$\hat{\delta}(\rho) = \max_{\omega_k} \left| \frac{\hat{\Phi}_{r\varepsilon}(\omega_k,\rho)}{\hat{\Phi}_r(\omega_k)} \right| \tag{24}$$

Note that a rectangular window is applied here, other windows can be used. Using the controller parameterization (11), $\hat{\Phi}_{r\varepsilon}(\omega_k,\rho)$ can be expressed as a linear combination of the controller parameters:

$$\hat{\Phi}_{r\varepsilon}(\omega_k,\rho) = \frac{1}{N}\sum_{\tau=-l_2}^{l_2}\sum_{t=1}^{N} \left[r(t-\tau)Mr(t)e^{-j\tau\omega_k} - r(t-\tau)\beta^T(1-M)y(t)e^{-j\tau\omega_k}\rho \right] \tag{25}$$

The estimate (24) can be used to define a set of convex constraints such that $\hat{\hat{\delta}}(\rho) \leqslant \delta_N$. With these constraints (14) gives the following convex optimization problem:

$$\begin{aligned} \hat{\rho} = \arg \min_{\rho \in \mathcal{D}_K} & \; J_{N,l_1}(\rho) \\ \text{subject to} & \\ \left| \sum_{\tau=-l_2}^{l_2} \hat{R}_{r\varepsilon}(\tau,\rho) e^{-j\tau\omega_k} \right| \leqslant \delta_N & \left| \sum_{\tau=-l_2}^{l_2} \hat{R}_r(\tau) e^{-j\tau\omega_k} \right|, \\ \omega_k = & 2\pi k/(2l_2+1), \quad k = 0,\ldots,l_2+1 \end{aligned} \tag{26}$$

This problem can be solved for up to several thousand constraints and the solution is the global optimum.

Theorem 2. *Consider the controller structure defined in (11). Let the filter W be defined as:*

$$W(e^{-j\omega}) = \frac{F(e^{-j\omega})(1-M(e^{-j\omega}))}{\Phi_r(\omega)} \tag{27}$$

This filter might be non-causal. Assume that ***A1-A4*** *are satisfied, that W and $(1-M)G$ have no zero on the imaginary axis and that a strictly feasible solution exists for (26), for the series of optimization problems as $N, l_1, l_2 \to \infty$ as well as for (14). Then, as $N, l_1, l_2 \to \infty$ and $l_1/N, l_2/N \to 0$, the optimizer $\hat{\rho}$ in (26) converges w.p.1 to the stabilizing optimizer of $J(\rho)$ defined in (14):*

$$\lim_{N,l_1,l_2\to\infty, l_1/N, l_2/N\to 0} \hat{\rho} = \rho_s, \tag{28}$$

Proof: The proof is given in Appendix I. ■

4.2. Implementation for nonminimum-phase or unstable systems

Let the unstable or minimum-phase plant G be excited by $r(t)$ in closed loop according to the scheme of Fig. 3. The output of the plant is affected by the noise $v(t)$. The discrete signals $r(t)$, $y(t)$, $u_1(t)$ and $u_2(t)$ of length N are available. The error $\varepsilon(t,\rho)$ is given by (16). The error signal $\varepsilon_s(t,\rho)$ used in the stability constraints is given by (17). Optimization problem (10) can be approximated by the following convex optimization problem:

$$\begin{aligned} \hat{\rho} = \arg \min_{\rho \in \mathcal{D}_K} & \; J_{N,l_1}(\rho) \\ \text{subject to} & \\ \left| \sum_{\tau=-l_2}^{l_2} \hat{R}_{r\varepsilon_s}(\tau,\rho) e^{-j\tau\omega_k} \right| \leqslant \delta_N & \left| \sum_{\tau=-l_2}^{l_2} \hat{R}_r(\tau) e^{-j\tau\omega_k} \right| \end{aligned} \tag{29}$$

where $\hat{R}_r(\tau)$ is defined in (23) and $\hat{R}_{r\varepsilon_s}(\tau,\rho)$ is given by:

$$\hat{R}_{r\varepsilon_s}(\tau,\rho) = \frac{1}{N}\sum_{t=1}^{N} r(t-\tau)\varepsilon_s(t,\rho), \quad \tau = -l_2,\ldots,l_2.$$

Theorem 3. *Consider the controller structure defined in (11). Let the filter W be defined as:*

$$W(e^{-j\omega}) = \frac{F(e^{-j\omega})(1 - M(e^{-j\omega}))}{\big(1 - M_s(e^{-j\omega})\big)\Phi_r(\omega)} \tag{30}$$

*Assume that **A1-A4** are satisfied, that W and $(1-M)G/(1+K_sG)$ has no zero on the imaginary axis and that a strictly feasible solution exists for (29), for the series of optimization problems as $N, l_1, l_2 \to \infty$ as well as for (10). Then, as $N, l_1, l_2 \to \infty$ and $l_1/N, l_2/N \to 0$, the optimizer $\hat{\rho}$ in (29) converges w.p.1 to the stabilizing optimizer $J(\rho)$ as defined in (10):*

$$\lim_{N,l_1,l_2\to\infty, l_1/N, l_2/N\to 0} \hat{\rho} = \rho_s \tag{31}$$

Proof: The proof is given in Appendix II ■

Remark: The filter W depends on the unknown plant G and thus cannot be implemented. However,

$$\big(1 - M_s(e^{-j\omega})\big)\Phi_r(\omega) = \frac{1}{1 + K_s(e^{-j\omega})G(e^{-j\omega})}\Phi_r(\omega) = \Phi_{ru_2}(\omega), \tag{32}$$

where $\Phi_{ru_2}(\omega)$ is the cross-spectrum between $r(t)$ and $u_2(t)$, which can be estimated using the measured data. The weighting filter is then given by:

$$W(e^{-j\omega}) = \frac{F(e^{-j\omega})(1 - M(e^{-j\omega}))}{\Phi_{ru_2}(\omega)}. \tag{33}$$

4.3. Using a finite number of data

The following analysis is detailed for the scheme for stable minimum-phase plants.

Asymptotically, the data-driven method proposed in Section 4.1 leads to a stabilizing controller which, according to Theorem 2, solves (14). In practice, only a finite number of data is available and an approximation of (14) is used. The quality of the approximation of the control criterion is analyzed next. The quality of the estimate of the stability constraint is discussed at the end of this section.

Approximating the control criterion

Using assumption **A3**, the error $\varepsilon(t, \rho)$ can be written as:

$$\begin{aligned}\varepsilon(t,\rho) &= \big[M - K(\rho)(1-M)G\big]r(t) - K(\rho)(1-M)He(t) \\ &= Dr(t) - Le(t) = r_D(t) - e_L(t)\end{aligned} \tag{34}$$

with obvious definitions for the filters D and L. $r_D(t)$ represents the part of the error stemming from the reference signal $r(t)$, and $e_L(t)$ results from the stochastic noise $v(t) = He(t)$. The correlation function $f_{N,l_1}(\rho)$ can be expressed as:

$$f_{N,l_1}(\rho) = \frac{1}{N}\sum_{t=1}^{N} \zeta(t)\big[r_D(t) - e_L(t)\big] \tag{35}$$

In the absence of noise, the correlation criterion is given by:

$$\tilde{J}_{N,l_1}(\rho) = \frac{1}{N^2}\sum_{t=1}^{N}\zeta^T(t)r_D(t)\sum_{t=1}^{N}\zeta(t)r_D(t) = \sum_{\tau=-l_1}^{l_1}\hat{R}^2_{r_W r_D}(\tau) \tag{36}$$

where $\hat{R}^2_{r_W r_D}(\tau)$ is an estimate of the cross-correlation between $r_W(t)$ and $r_D(t)$. The length of $\zeta(t)$ defines the size of the rectangular window. The expected value of the correlation criterion $J_{N,l_1}(\rho)$ based on a finite number of data can then be expressed as:

$$E\left\{J_{N,l_1}(\rho)\right\} \approx \tilde{J}_{N,l_1}(\rho) + \frac{\sigma^2(2l_1+1)}{2\pi N}\int_{-\pi}^{\pi}\frac{|1-M|^4|K(\rho)|^2|H|^2|F|^2}{\Phi_r(\omega)}d\omega, \quad (37)$$

where the expected value is taken with respect to the noise $e(t)$. Consequently, the minimizer $\hat{\rho}$ of $J_{N,l_1}(\rho)$ based on a finite number of data is biased. In Appendix III it is shown how (37) is derived.

Asymptotically, $\tilde{J}_{N,l_1}(\rho)$ converges to $J(\rho)$ and the second term becomes zero, thus corresponding to the result of Theorem 2. However, for a finite number of data, the deterministic $\tilde{J}_{N,l_1}(\rho)$ leads to a windowed estimate of $J(\rho)$ and the second term adds a bias to the minimizer of this estimate.

Remarks:

- The controller that minimizes the biased criterion $J_{N,l_1}(\rho)$ will have a low gain wherever $|1-M|^2|H||F|$ is large. $(1-M)$ is the sensitivity function of the reference model and H represents the frequency contents of the noise. Hence, the controller gain is reduced at frequencies where both the sensitivity and the noise are high. This will in general increase the robustness of the closed-loop system.
- The controller gain is reduced in the frequency ranges where the input spectrum is weak. This is an interesting characteristic in the sense that, if the data is not informative in a frequency region, the controller gain in this region is decreased, which again increases the robustness of the closed-loop system.
- The bias in $J_{N,l_1}(\rho)$ decreases as the number of data N increases. It increases as the number of lags l_1 used in the instrumental variable vector $\zeta(t)$ increases.

Practical issues

The choice of l_1 determines the quality of the estimate $\tilde{J}_{N,l_1}(\rho)$. Assume that $R_{r_W r_D}(\tau) \approx 0$ for $|\tau| > \tau_0$, where τ_0 is an integer that depends on the length of the impulse response of $W(q^{-1})D(q)$ and the length of $R_r(\tau)$. In order to find a good estimate of $J(\rho)$, the length l_1 of $\zeta(t)$ should be chosen as $l_1 \geqslant \tau_0$. However, (37) states that the bias increases as l_1 increases. With the choice of l_1 a trade-off is made between accuracy and bias.

Estimating the stability constraint

According to Theorem 1, $\delta(\rho) < 1$ is sufficient for closed-loop stability. In practice, only the estimate $\hat{\delta}(\rho)$ of $\delta(\rho)$ is available and the stability constraint is no longer sufficient. The reliability of the stability constraint depends on the quality of the estimate. If additional information regarding the plant and measurement noise is available, bounds on the estimation error can be used to define an appropriate δ_N. If no additional assumptions are made, a decrease of δ_N will increase the reliability of the stability constraint, but also increase the conservatism. The reliability of the approach is thus comparable to a model-based approach, where modeling errors need to be taken into account in order to guarantee stability.

4.4. Implementation using periodic data

It is well known that the quality of spectral estimates can be improved when periodic data are used. For periodic data, the estimate does not contain leakage errors and has a decreasing variance with increasing number of periods [24]. Periodic excitation should therefore be used whenever possible. The use of periodic data also improves the quality of the correlation criterion. The trade-off for this improved quality is a limited frequency resolution. Only the case of stable, minimum-phase systems is summarized here. For periodic data, Assumptions **A1-A2** need to be replaced by:

A5 The reference signal is periodic with period T, i.e. $r(t+nT) = r(t)$ for any integer n. The auto-correlation of $r(t)$ is given by

$$R_r(\tau) = \frac{1}{T}\sum_{t=1}^{T} r(t-\tau)r(t), \tag{38}$$

for $\tau = 0, \ldots, T-1$. The signal $r(t)$ includes an integer number of periods, i.e. $N = n_p T$, with n_p the number of periods. The corresponding output of the plant is also periodic, i.e. there are no transients present in the response of the system.

A6 The spectrum of the periodic reference signal $r(t)$ satisfies

$$\Phi_r(\omega_k) = \sum_{\tau=0}^{T-1} R_r(\tau)e^{-j\tau\omega_k} \neq 0, \omega_k = 2\pi k/T, k = 0, \ldots, T-1 \tag{39}$$

For the periodic reference signal $r(t)$, the vector of instrumental variables defined in (20) is also periodic and its length should satisfy $l_1 \leqslant T/2$. The optimization problem (14) can then be approximated by:

$$\begin{gathered}\hat{\rho} = \arg\min_{\rho\in\mathcal{D}_K} J_{N,l_1}(\rho) \\ \text{subject to} \\ \left|\sum_{\tau=0}^{T-1} \hat{R}_{r\varepsilon}(\tau,\rho)e^{-j\tau\omega_k}\right| \leqslant \delta_N \left|\sum_{\tau=0}^{T-1} R_r(\tau)e^{-j\tau\omega_k}\right|, \\ \omega_k = 2\pi k/T, \quad k = 0, \ldots, \lfloor (T-1)/2 \rfloor\end{gathered} \tag{40}$$

Let the weighting filter W be chosen as

$$W(e^{-j\omega_k}) = \frac{F(e^{-j\omega_k})(1-M(e^{-j\omega_k}))}{\Phi_r(\omega_k)}. \tag{41}$$

Note that, in the periodic case, W is defined only for the frequencies ω_k where the spectrum $\Phi_r(\omega_k)$ is nonzero. It then follows from Theorem 2, with assumptions **A1-A2** replaced by **A5-A6**, that the optimizer of (40) converges to the stabilizing optimizer of $J(\rho)$ defined in (14):

$$\lim_{N,T\to\infty,T/N\to 0} \hat{\rho} = \rho_s, \text{w.p.1.} \tag{42}$$

Remark: If a parametric representation of $\Phi_r(\omega_k)$ is available, the filter W can be implemented in the time domain since $F(q^{-1})$ and $M(q^{-1})$ are known. If such a representation is not available, the exact filter (41) can be applied in the frequency domain. The periodic instrumental variables $\zeta(t)$ can therefore be found without any approximation, which is not the case for non-periodic reference signals. A bias expression similar to (37) can be found for the periodic case.

5. ILLUSTRATIVE EXAMPLES

5.1. Numerical example

Consider the plant given by the discrete-time model $G(q^{-1})$:

$$G(q^{-1}) = \frac{0.7893q^{-3}}{1 - 1.418q^{-1} + 1.59q^{-2} - 1.316q^{-3} + 0.886q^{-4}},$$

which corresponds to a stable minimum-phase model of the flexible transmission system proposed as a benchmark for digital control design in [25]. The control objective is defined by the reference model

$$M(q^{-1}) = \frac{q^{-3}(1-\alpha)^2}{(1-\alpha q^{-1})^2},$$

with $\alpha = 0.606$. The integral controller

$$K(\rho) = \frac{\rho_0 + \rho_1 q^{-1} + \rho_2 q^{-2} + \rho_3 q^{-3} + \rho_4 q^{-4} + \rho_5 q^{-5}}{1 - q^{-1}}$$

is chosen, with the unknown parameters $\rho_0, \ldots, \rho_5$. The reference model M has unity static gain, thus ensuring that $1 - M$ has a zero at 1, which makes Lemma 1 applicable.

CbT-GS

A PRBS signal of 255 samples with unity amplitude is used as input to the system. Four periods of this signal are used for controller design, $N = n_p T = 1020$. The periodic output is disturbed by a zero-mean white noise such that the signal-to-noise ratio is about 10 in terms of variance. The instrumental variables are defined according to (20), with $l_1 = 20$ in order to limit the bias due to the finite number of data. For the same reason, the bound in the stability condition is fixed to $\delta_N = 0.95$. Since the spectrum of the PRBS reference signal is known, $\Phi_r(\omega_k) = 1$, the weighting filter is implemented in the time domain; $F = 1$ and $W = 1 - M$. The constraints are implemented as in (40). A Monte Carlo simulation with 100 experiments is performed, using a different noise realization for each experiment.

Bode plots of the resulting closed-loop system for all 100 controllers are shown in Fig. 4. All 100 controllers stabilize the system and achieve acceptable performance. The stability constraint is active for 4 controllers; however, the difference between the unconstrained and the constrained solution is small. A small bias at high frequencies can be observed as expected from (37). Since the reference model is chosen appropriately, the optimal controller minimizing $J(\rho)$ stabilizes the system. Furthermore, because the quality of the estimate found using the correlation approach is good, the addition of the stability constraints does not affect the results.

Guaranteeing stability for VRFT

To show the effectiveness of the stability constraints, the same data are used to calculate controllers using the VRFT approach [4]. The goal is to show that, when the unconstrained problem has a destabilizing solution, addition of the stability constraints leads to stabilizing controllers. The VRFT approach that uses a second experiment to define the instrumental variables is used specifically to find these destabilizing controllers. This approach leads to an unbiased estimate, but it is well known that the use of noise-corrupted instrumental variables increases the variance of the estimate [26]. This variance might lead to instability even in the

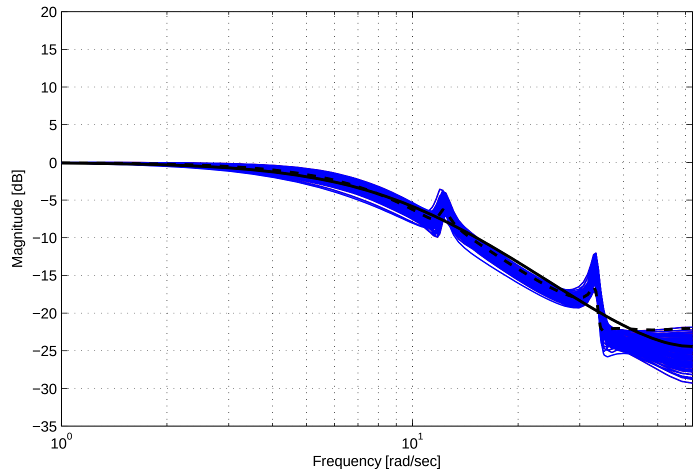

Figure 4. Magnitude Bode plots of M (thick line), achieved closed-loop performance in Monte Carlo simulation for CbT-GS (blue lines), and in the noise-free case (dashed line).

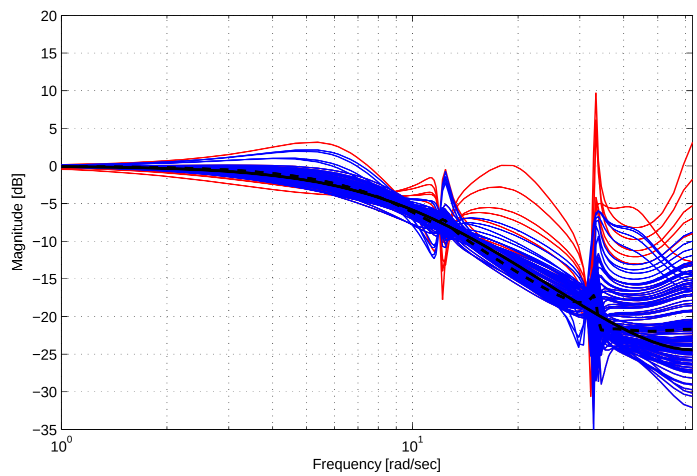

Figure 5. Magnitude Bode plots of M (thick line), achieved closed-loop performance without stability constraints for the 96 stabilizing VRFT controllers (red lines), with stability constraints for 100 stabilizing controllers (blue lines), and the noise-free case (dashed line).

case of an appropriate reference model. It should be noted that this variance results from the choice of instrumental variables and is not inherent to VRFT. Other methods to deal with measurement noise are suggested in [4].

For each of the 100 simulations, a second experiment is simulated with a different noise realization. Hence, the VRFT controllers are calculated using 2040 samples. Two controllers are calculated for each set of data. The first controller is calculated using the VRFT approach as proposed in [4]. For the second controller, the stability constraints are added to the VRFT problem. The samples available from both experiments are used in the constraints that are

Figure 6. Torsional setup, ECP Model 205

implemented as in (40).

Four of the controllers calculated using the unconstrained VRFT approach destabilize the system. All controllers calculated with the stability constraints stabilize the system. Note that, due to the conservatism in the stability criterion, 7 of the 96 stabilizing VRFT controllers would not satisfy the stability constraints. The optimum of the constrained optimization problem is therefore different than the VRFT solution. For these stabilizing controllers, the active constraints indicate poor closed-loop performance and the conservatism in the constraints actually leads to better performance. This can be seen in Fig. 5, which shows the magnitude Bode plots of all stabilizing controllers (96 for the unconstrained problem and 100 for the constrained problem).

Since only an estimate of the stability constraint is used and the estimation error is not taken into account, stability cannot be guaranteed theoretically. However, all of the 100 controllers do stabilize the system. It is shown in [27] that the H_∞-norm is in general overestimated if the data is affected by noise. In this example, the DFT estimate does indeed overestimate the H_∞-norm for all of the 100 noise realizations. The norm is overestimated by at least 0.08 and by at the most 0.73. The average overestimation is 0.39. Even though only an estimate of the stability constraint is used, stability is guaranteed for each of the 100 controllers.

5.2. Control of an experimental torsional setup

The effectiveness of CbT-GS is demonstrated experimentally on the torsional setup shown in Fig. 6. The setup consists of three discs connected by a torsionally flexible shaft. Two masses are fixed to each disc. The shaft is driven by a brushless servo motor. The angular displacement of the top disc is measured by an encoder and expressed in degrees. The plant is minimum phase, contains an integrator and has two strong resonances. The sampling time is 60 ms.

A set of periodic open-loop data is collected using a zero-mean PRBS input of 255 samples. Five periods of input and output measurements are used for controller design. The controller

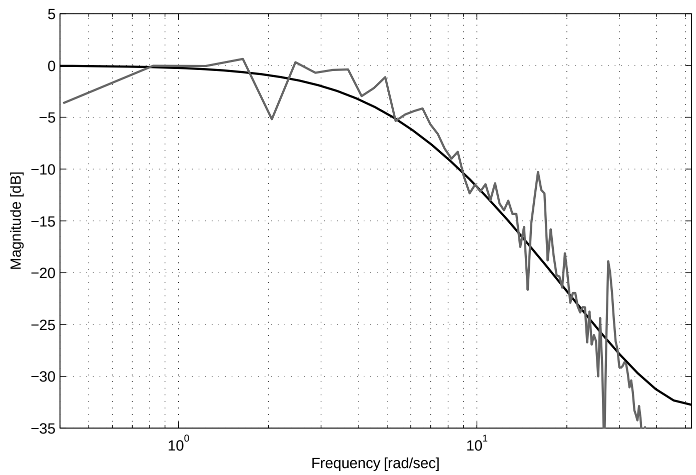

Figure 7. Magnitude Bode plot of the reference model M_1 (black) and the estimated closed-loop plant controlled by K_1 (grey).

structure is fixed as a 7th-order FIR filter. The controllers are calculated using (40) with $F = 1$ and $l_1 = 127$. The bound in the stability condition is fixed as $\delta_N = 0.8$. The reference model needs to have unity static gain since the plant contains an integrator. Two different reference models are considered. The first one reads:

$$M_1 = \frac{0.0765q^{-1}}{(1-0.7q^{-1})^2(1-0.15q^{-1})}.$$

The second reference model is chosen with similar bandwidth but a high-frequency roll-off of only one:

$$M_2 = \frac{0.3q^{-1}}{1-0.7q^{-1}}.$$

The stability constraints in the optimization problem for M_1 are not active, the resulting controller is denoted K_1. In contrast, the stability constraints are active in the optimization problem for M_2. Two controllers are calculated using M_2: controller K_2 is the unconstrained optimum, controller K_3 is the solution to the constrained problem.

When applied to the plant, controller K_2 leads to instability. Stability is obtained with K_1 and K_3, for which the closed-loop frequency-response can be identified. Four periods of the PRBS of 255 samples with amplitude 50 degrees are collected on the plant controlled by K_1. The frequency response estimated using DFT is shown in Fig. 7. The reference model M_1 is appropriate, and the achieved closed-loop system resembles the reference model. The steady-state gain is smaller than one due to static friction. The plant controlled by K_3 is excited by a PRBS with a frequency divider of 2, 510 samples per period and amplitude 50 degrees. Three periods are used for the DFT estimate. The result is shown in Fig. 8. The controller does stabilize the plant but the required closed-loop performance is not achieved. Reference model M_2 is inappropriate and cannot be achieved. Since the stability constraints are active, CbT-GS actually indicates this problem.

Remark: In the numerical example of Section 5.1, addition of the stability constraints to VRFT improves the closed-loop performance. The reference model is appropriate and

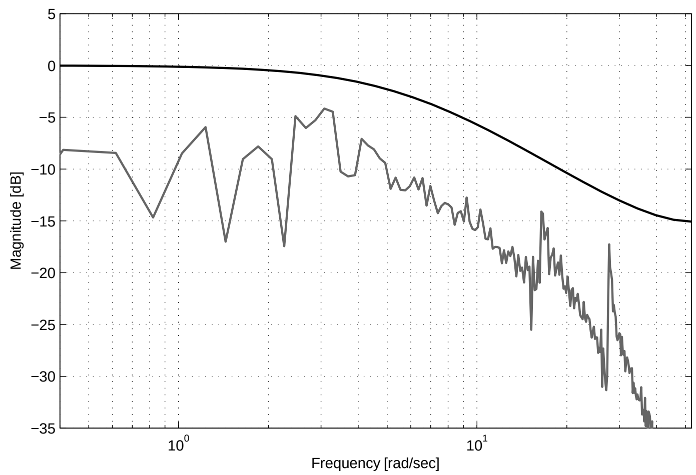

Figure 8. Magnitude Bode plot of the reference model M_2 (black) and the estimated closed-loop plant controlled by K_3 (grey).

instability is the result of the variance of the estimated controller parameters. For the experimental torsional setup, instability is due to an inappropriate reference model. Addition of the stability constraints leads to a stabilizing solution, but the closed-loop performance remains poor because it is not possible to achieve the required performance.

6. CONCLUSIONS

Possible instability of the closed-loop system is one of the main difficulties in data-driven controller tuning. The stability constraints introduced in this paper guarantee a stabilizing solution as the number of data tends to infinity. Other difficulties in data-driven controller tuning include the necessity of many experiments, the effect of measurement noise on the controller parameters, and the appropriateness of the control objective with respect to the plant characteristics and the controller structure. CbT-GS as presented in this paper is an attempt to solve these problems using convex optimization.

Only one experiment in open- or closed-loop operation is required to approximate the model reference control criterion. The correlation criterion and the sufficient condition for closed-loop stability can be represented by convex functions of linearly parameterized controllers. The quality of the resulting controller can be assessed by looking at the correlation between the residuals and the reference signal. Furthermore, if the constraints in the optimization problem are active, this indicates that the control objective cannot be achieved by the chosen controller structure. This suggests increasing the controller order or modifying the reference model by an iterative procedure that can be performed off-line without additional experiments.

The proposed constrained model-reference control problem could also be solved using a model-based approach. In this case, many well-known techniques for analysis and performance evaluation can be used. Furthermore, the main results presented in this work are asymptotic in the number of data, and asymptotically, the result is equivalent to a model-based approach.

So why would one prefer the proposed data-driven approach over a model-based solution? Firstly, model identification and controller design are combined in the proposed approach. The asymptotic results of the model identification, which are hidden in a model-based controller design approach, become apparent in such a data-to-controller procedure. Analogous to the model-based approach, the results can be extended to finite data length if the modeling errors are taken into account. Secondly, the advantages of a data-driven approach are apparent in practice. In the data-driven approach, no plant model is identified and the problem of undermodeling is avoided. Furthermore, since the controller parameters are a non-linear function of the estimated model, the controller estimate will be biased with respect to noise and a direct approach can achieve higher accuracy. Unfortunately, current tools for accuracy analysis do not allow for a complete comparison of model-based and data-driven approaches.

APPENDIX

I. PROOF OF THEOREM 2

Firstly, stochastic convergence of the unconstrained problem is established. We have [24]:

$$\lim_{N\to\infty} f_{N,l_1}(\rho) = [R_{r_W\varepsilon}(-l_1,\rho),\ldots,R_{r_W\varepsilon}(l_1,\rho)]^T, \quad \text{w.p. 1}$$

The correlation criterion is a continuous function of this variable, which leads to ([28], p. 450):

$$\lim_{N\to\infty} J_{N,l_1}(\rho) = \sum_{\tau=-l_1}^{l_1} R^2_{r_W\varepsilon}(\tau,\rho), \text{ w.p. 1.} \tag{43}$$

Note that this result holds for finite l_1. In this case, the correlation criterion converges because $N\to\infty$ implies $l_1/N\to 0$.

Secondly, convergence of this deterministic variable to $J(\rho)$ is established as $l_1\to\infty$. Since $\Delta(\rho)$ is stable, $\sum_{\tau=-l_1}^{l_1} R^2_{r_W\varepsilon}(\tau,\rho)$ and the limit $\sum_{\tau=-\infty}^{\infty} R^2_{r_W\varepsilon}(\tau,\rho)$ are bounded on $\mathcal{D}_K$. The sequence of deterministic convex functions $\sum_{\tau=-l_1}^{l_1} R^2_{r_W\varepsilon}(\tau,\rho)$ then converges uniformly to $\sum_{\tau=-\infty}^{\infty} R^2_{r_W\varepsilon}(\tau,\rho)$ on the compact set $\mathcal{D}_K$ as $l_1\to\infty$. This follows from Theorem 10.8 in [29], which states that pointwise convergence of a series of convex functions to a convex limit function implies uniform convergence on a compact set.

It then follows that, as $N, l_1\to\infty, l_1/N\to 0$, the correlation criterion converges uniformly:

$$\lim_{N,l_1\to\infty,l_1/N\to 0} J_{N,l_1}(\rho) = \sum_{\tau=-\infty}^{\infty} R^2_{r_W\varepsilon}(\tau,\rho), \text{ w.p. 1.} \tag{44}$$

Using Parseval's theorem, this is equivalent to:

$$\sum_{\tau=-\infty}^{\infty} R^2_{r_W\varepsilon}(\tau,\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Phi_{r_W\varepsilon}(\omega,\rho)|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |W\big(M-K(\rho)(1-M)G\big)|^2\Phi_r^2(\omega)d\omega$$

With the expression of W given in (27), (44) becomes:

$$\lim_{N,l_1\to\infty,l_1/N\to 0} J_{N,l_1}(\rho) = J(\rho), \text{w.p.1.} \tag{45}$$

Convergence of the constraints is shown next. As $N \to \infty, l_2/N \to 0$, the estimate $\hat{R}_{r\varepsilon}(\tau,\rho)$ converges w.p.1 to $R_{r\varepsilon}(\tau,\rho)$ and $\hat{R}_r(\tau)$ converges w.p.1 to $R_r(\tau)$, for $\tau = [-l_2,\ldots,l_2]$. Consequently, $\frac{\hat{\Phi}_{r\varepsilon}(\omega_k,\rho)}{\hat{\Phi}_r(\omega_k)}$ converges pointwise to $\Delta(\omega_k)$, w.p.1. $\Delta(\omega_k)$ and $\delta(\rho)$ are bounded on $\mathcal{D}_K$ since Δ is stable. The series of convex functions $\max_{\omega_k}|\Delta(\omega_k)|$ then converges uniformly to the convex function $\delta(\rho)$ as $l_2 \to \infty$ (Theorem 10.8 of [29]). It follows that, with probability 1, $\max_{\omega_k}\left|\frac{\hat{\Phi}_{r\varepsilon}(\omega_k,\rho)}{\hat{\Phi}_r(\omega_k)}\right|$ converges uniformly to $\delta(\rho)$ as $N, l_2 \to \infty, l_2/N \to 0$.

Convergence of the constrained optimization then follows from the dual problem (Theorem 1.44 [30]): Consider the function $\mathcal{L}(\rho) := J(\rho)+\nu(\delta(\rho)-\delta_N)$, where ν is the Lagrange multiplier and (ν_0,ρ_0) is a KKT point of $\mathcal{L}(\rho)$. Then ρ_0 is the global optimizer of (14). Since $J_{N,l_1}(\rho)$ and $\max_{\omega_k}\left|\frac{\hat{\Phi}_{r\varepsilon}(\omega_k,\rho)}{\hat{\Phi}_r(\omega_k)}\right|$ converge uniformly to $J(\rho)$ and $\delta(\rho)$, the dual of (26) converges uniformly to $\mathcal{L}(\rho)$. Since the convergence is uniform, it follows that the optimizer of (26) converges to the optimizer of (14).

II. PROOF OF THEOREM 3

The proof is similar to that of Theorem 2. Even though G might be unstable, the filter $(1-M_s)G$ is stable and consequently all filters involved are stable. As $N, l_1 \to \infty$ and $l_1/N \to 0$, the correlation function $f_{N,l_1}(\rho)$ converges to the cross-correlation between $r_W(t)$ and $\varepsilon(t,\rho)$;

$$R_{r_W\varepsilon}(\tau,\rho) = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}E\left\{r_W(t-\tau)\varepsilon(t,\rho)\right\}$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}r_W(t-\tau)(1-M_s)\big[M-K(\rho)(1-M)G\big]r(t).$$

Using Parseval's theorem, the correlation criterion converges to:

$$\sum_{\tau=-\infty}^{\infty}R^2_{r_W\varepsilon}(\tau,\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi}|\Phi_{r_W\varepsilon}(\omega,\rho)|^2 d\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}|W(1-M_s)\big[M-K(\rho)(1-M)G\big]|^2\Phi_r^2(\omega)d\omega$$

Using the expression of W in (30) leads to

$$\lim_{N,l_1\to\infty,l_1/N\to 0} J_{N,l_1}(\rho) = J(\rho) \tag{46}$$

The rest of the proof follows that of Theorem 2.

III. PROOF OF (37)

$e_L(t)$ can be written as

$$e_L(t) = \sum_{k=0}^{\infty} l_k e(t-k),$$

with l_k the impulse response of L. The vector of random variables:

$$X_N = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \zeta(t) e_L(t)$$

converges in distribution to a normal distribution with zero mean and variance P [24]:

$$P = \lim_{N\to\infty} E\left\{X_N X_N^T\right\} = \sigma^2 \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E\left\{\tilde{\zeta}(t)\tilde{\zeta}^T(t)\right\},$$

where

$$\tilde{\zeta}(t) = \sum_{k=0}^{\infty} l_k \zeta(t+k) = L(q)W(q^{-1})[r(t+l_1), r(t+l_1-1), \ldots r(t), r(t-1), \ldots, r(t-l_1)]^T \quad (47)$$

The diagonal elements of P are equal to $\sigma^2 R_{r_{LW}}(0)$, where $R_{r_{LW}}(\tau)$ is the auto-correlation function of $L(q)W(q^{-1})r(t)$. The expected value $E\{J_{N,l_1}(\rho)\}$ can then be expressed as:

$$\begin{aligned} E\{J_{N,l_1}(\rho)\} &= E\left\{\frac{1}{N^2} \sum_{t=1}^{N} \zeta^T(t)[r_D(t) - e_L(t)] \sum_{s=1}^{N} \zeta(s)[r_D(s) - e_L(s)]\right\} \\ &= E\left\{\frac{1}{N^2} \sum_{t=1}^{N} \zeta^T(t) r_D(t) \sum_{s=1}^{N} \zeta(s) r_D(s)\right\} - 2E\left\{\frac{1}{N^2} \sum_{t=1}^{N} \zeta^T(t) r_D(t) \sum_{s=1}^{N} \zeta(s) e_L(s)\right\} \\ &\quad + E\left\{\frac{1}{N^2} \sum_{t=1}^{N} \zeta^T(t) e_L(t) \sum_{s=1}^{N} \zeta(s) e_L(s)\right\} = \tilde{J}_{N,l_1}(\rho) - 0 + \frac{1}{N} E\left\{X_N^T X_N\right\} \quad (48) \end{aligned}$$

For large N, the distribution of X_N is well approximated by P, and $E\{J_{N,l_1}(\rho)\}$ can be approximated using this asymptotic distribution:

$$\begin{aligned} E\{J_{N,l_1}(\rho)\} = \tilde{J}_{N,l_1}(\rho) + \frac{1}{N} E\left\{X_N^T X_N\right\} &\approx \tilde{J}_{N,l_1}(\rho) + \frac{1}{N}\mathrm{trace}(P) \\ &= \tilde{J}_{N,l_1}(\rho) + \frac{2l_1+1}{N} \sigma^2 R_{r_{LW}}(0) \quad (49) \end{aligned}$$

Using Parseval's theorem, this can be expressed as:

$$\begin{aligned} E\{J_{N,l_1}(\rho)\} &\approx \tilde{J}_{N,l_1}(\rho) + \frac{2l_1+1}{N} \sigma^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{r_{LW}}(\omega) d\omega \\ &= \tilde{J}_{N,l_1}(\rho) + \frac{\sigma^2(2l_1+1)}{2\pi N} \int_{-\pi}^{\pi} |L(e^{-j\omega}) W(e^{-j\omega})|^2 \Phi_r(\omega) d\omega \end{aligned}$$

Replacing L by $(1-M)K(\rho)H$ and W by $F(1-M)/\Phi_r$ gives (37).

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