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S. Brunner and J. Vaclavik

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S. Brunner and J. Vaclavik

Centre de Recherches en Physique des Plasmas

Association Euratom - Confédération Suisse

Ecole Polytechnique Fédérale de Lausanne

21, av. des Bains – CH-1007 Lausanne/Switzerland

Abstract

The drift kinetic equation (DKE) is used to establish a formula for power absorption of small amplitude, low frequency electromagnetic (EM) fields in a hot toroidal axisymmetric plasma. The stationary plasma is first considered. Electrons and ions are described by local Maxwellian distributions, alpha particles by a local slowing-down distribution. The fluctuating part of the distribution function for each species is then evaluated from the linearized DKE in terms of the EM fields using a perturbation method. The parameter $b_p = B_p/| \vec{B}_0 |$, where B_p is the poloidal component of the magnetostatic field $\vec{B}_0$, and the parameter $| \vec{v_d} |/\lambda_\perp \omega$, where $\vec{v_d}$ is the magnetic curvature drift, $\lambda_\perp$ the wavelength perpendicular to $\vec{B}_0$ and ω the frequency of the EM fields, are considered to be small. By integrating the resulting distribution function over velocity space, an explicit formula for the power absorbed by each species is obtained. To obtain an expression suitable for direct implementation in an ideal-MHD code, the electric field component parallel to the magnetostatic field is evaluated using the quasi-neutrality equation.

I. Introduction

In this paper we will present a derivation of a formula for the power absorbed by hot tokamak plasmas for given low frequency electromagnetic (EM) fields. This study follows the calculation of the dielectric tensor of such plasmas[1]. This work is complementary, as we restrict ourselves to low frequency EM fields, which allows us to go beyond the results obtained for the tensor. For perturbation frequencies much smaller than the cyclotron frequency and $\rho/L \ll 1$, which compares the characteristic length L of all dynamical variables with the Larmor radius ρ, it is suitable to describe the evolution of the corresponding species using the drift kinetic equation (DKE) instead of the full Vlasov equation. Using this model, the distribution function of a stationary toroidal axisymmetric plasma is first evaluated. The perturbation by the EM fields is then considered. This solution to the DKE is presented in Sec.II. The total power, averaged over time, is evaluated in Sec.III. If the EM fields are provided by an ideal-MHD code, the electric field component parallel to the magnetostatic field is zero and it must therefore be evaluated. This is done iteratively in Sec.IV using the quasi-neutrality equation. Conclusions are drawn in Sec.V.

II. Solution of the DKE

The system we consider is a stationary plasma confined by a magnetostatic field $\vec{B}_0$. This plasma is then perturbed by EM fields $(\vec{E}, \vec{\tilde{B}})$. Let us define the ordering applied for the following derivations in terms of a small parameter λ. The perturbation frequency ω is assumed to be much smaller than the cyclotron frequency $\Omega = qB_0/m$, so that

$$\frac{\omega}{\Omega} = \mathcal{O}(\lambda), \qquad (1)$$

and

$$\frac{\rho}{L} = \mathcal{O}(\lambda). \qquad (2)$$

Furthermore, the electric force is considered small as compared with the magnetic force

$$\frac{|\vec{E}|}{v_{th}|\vec{B}_0|} = \mathcal{O}(\lambda), \tag{3}$$

where $v_{th}^2 = 2T/m$ is the thermal velocity squared. Faraday's law and relations (1), (2) and (3) also imply

$$\frac{|\vec{\tilde{B}}|}{|\vec{B}_0|} = \mathcal{O}(\lambda). \tag{4}$$

A distribution function f can be defined in the guiding center "phase space" $(\vec{X}, v_\perp, v_\parallel)$. The corresponding evolution equation is the DKE[2]:

$$\frac{Df}{Dt} = \left[\frac{\partial}{\partial t} + \vec{v}_g \cdot \frac{\partial}{\partial \vec{X}} + \frac{dv_\perp}{dt}\frac{\partial}{\partial v_\perp} + \frac{dv_\parallel}{dt}\frac{\partial}{\partial v_\parallel}\right] f = 0, \tag{5}$$

stating that the distribution function along a guiding center trajectory is constant. The flow in "phase space" is given consistently to order $\mathcal{O}(\lambda)$ by

$$\begin{aligned}
\frac{d\vec{X}}{dt} &= \vec{v}_g = v_\parallel \vec{e}_\parallel + \vec{v}_E + \frac{m}{qB}\vec{e}_\parallel \times \left[\frac{v_\perp^2}{2}\nabla \ln B + v_\parallel^2 \vec{e}_\parallel \cdot (\nabla \vec{e}_\parallel)\right], \\
\frac{dv_\perp}{dt} &= \frac{v_\perp}{2B}\left[(v_\parallel \vec{e}_\parallel + \vec{v}_E)\cdot \nabla B + \frac{\partial B}{\partial t}\right], \\
\frac{dv_\parallel}{dt} &= \frac{q}{m}E_\parallel + \frac{v_\perp^2}{2}\nabla \cdot \vec{e}_\parallel + v_\parallel \vec{e}_\parallel \cdot (\nabla \vec{e}_\parallel) \cdot \vec{v}_E,
\end{aligned} \tag{6}$$

where $\vec{B} = \vec{B}_0 + \vec{\tilde{B}}$ is the total magnetic field, $\vec{e}_\parallel = \vec{B}/B$ and $\vec{v}_E = \vec{E} \times \vec{B}/B^2$ the electric drift. In a complete study, the EM fields would have to be self consistent with the evolution of the plasma, however for the present work we consider them as given. Adopting the variables $(\vec{X}, \epsilon, \mu)$ where

$$\epsilon = \frac{1}{2}(v_\parallel^2 + v_\perp^2) \quad \text{and} \quad \mu = \frac{v_\perp^2}{2B_0}, \tag{7}$$

Eqs (5) and (6) are transformed to

$$\frac{Df}{Dt} = \left[\frac{\partial}{\partial t} + \vec{v}_g \cdot \frac{\partial}{\partial \vec{X}} + \frac{d\epsilon}{dt}\frac{\partial}{\partial \epsilon} + \frac{v_\perp}{B_0}\frac{dv_\perp}{dt}\frac{\partial}{\partial \mu}\right] f = 0, \tag{8}$$

$$\frac{d\epsilon}{dt} = \frac{q}{m}\vec{v}_g \cdot \vec{E} + \frac{v_\perp^2}{2B}\frac{\partial B}{\partial t}. \tag{9}$$

For a stationary state described by a distribution function f_0 and a magnetostatic field $\vec{B}_0$, Eq.(8) becomes

$$\vec{v}_{g0} \cdot \frac{\partial f_0}{\partial \vec{X}} = 0, \tag{10}$$

using the notations

$$\vec{v}_{g0} = v_\parallel \vec{e}_{\parallel 0} + \vec{v}_d \quad , \quad \vec{v}_d = \frac{1}{\Omega} \vec{e}_{\parallel 0} \times \left[\mu \nabla B_0 + v_\parallel^2 \vec{e}_{\parallel 0} \cdot (\nabla \vec{e}_{\parallel 0})\right] \text{ and } \vec{e}_{\parallel 0} = \frac{\vec{B}_0}{B_0}. \tag{11}$$

The quantity $\vec{v}_d$ is the drift due to the curvature and variation of amplitude of the magnetostatic field. Let us write f_0 in a series expansion with respect to λ

$$f_0 = F + F^{(1)} + \ldots \tag{12}$$

In zeroth order, Eq.(10) reduces to

$$v_\parallel \vec{e}_{\parallel 0} \cdot \frac{\partial F}{\partial \vec{X}} = 0. \tag{13}$$

From now on, we restrict ourselves to axisymmetric systems. In this case, the magnetostatic field can be written as follows

$$\vec{B}_0 = \nabla \psi \times \nabla \varphi + r B_{0\varphi} \nabla \varphi, \tag{14}$$

(r, φ, z) being the cylindrical coordinates. Relation(14) shows that ψ =const is a magnetic surface. It is convenient to adopt the space variables (ψ, χ, φ), where χ is defined such that the local unit vectors

$$\vec{e}_n = \frac{\nabla \psi}{|\nabla \psi|} \quad , \quad \vec{e}_p = \frac{\nabla \chi}{|\nabla \chi|} \text{ and } \vec{e}_\varphi = \frac{\nabla \varphi}{|\nabla \varphi|} \tag{15}$$

define an orthonormal system. The EM fields are projected onto the local magnetic orthonormal coordinate system $(\vec{e}_n, \vec{e}_b, \vec{e}_{\parallel 0})$. Using Eq.(13) and the axial symmetry, one can write

$$F = F(\psi, \epsilon, \mu). \tag{16}$$

For electrons and ions we take for F a local Maxwellian distribution

$$F = \frac{N(\psi)}{(\pi v_{th}^2(\psi))^{3/2}} \exp\left(-2\frac{\epsilon}{v_{th}^2(\psi)}\right), \tag{17}$$

N being the local density. The alpha particles are described by a slowing-down distribution[3]

$$F = N(\psi)\frac{C(\psi)}{v^3 + v_c^3(\psi)}H(v_0 - v), \quad (18)$$

$$v_c = \left(3\sqrt{\pi}\frac{m_\alpha + m_i}{m_\alpha m_i}m_e\right)^{1/3}\left(\frac{T_e}{m_e}\right)^{1/2}, \quad (19)$$

$$C = \frac{3}{4\pi \ln\left[\left(\frac{v_0}{v_c}\right)^3 + 1\right]}, \quad (20)$$

where H is the Heaviside function and $v_0 = 1.3 \cdot 10^7 m/s$ the birth velocity of the alphas. To order $\mathcal{O}(\lambda)$ Eq.(10) reads

$$v_\parallel \vec{e}_{\parallel 0} \cdot \frac{\partial F^{(1)}}{\partial \vec{X}} + \vec{v}_d \cdot \frac{\partial F}{\partial \vec{X}} = 0. \quad (21)$$

Solving this equation leads to

$$F^{(1)} = -\frac{v_\parallel}{\Omega b_p}\nabla_n F, \quad (22)$$

where $\vec{e}_{\parallel 0} \equiv b_p \vec{e}_p + b_\varphi \vec{e}_\varphi$.

We now consider the excitation of the stationary plasma by the EM fields $(\vec{E}, \vec{\tilde{B}})$. Let $\tilde{f}$ be the corresponding fluctuating part of the distribution function

$$f = f_0 + \tilde{f}. \quad (23)$$

For small excitations, one can linearize the DKE. This equation valid to order $\mathcal{O}(\lambda)$, is then given by

$$\left(\frac{\partial}{\partial t} + \vec{v}_{g0} \cdot \frac{\partial}{\partial \vec{X}}\right)\tilde{f} = -\left(\vec{\tilde{v}}_g \cdot \frac{\partial}{\partial \vec{X}} + \frac{\tilde{d\epsilon}}{dt}\frac{\partial}{\partial \epsilon}\right) f_0, \quad (24)$$

with the notations

$$\vec{\tilde{v}}_g = v_\parallel \vec{\tilde{e}}_\parallel + \frac{\vec{E} \times \vec{B}_0}{B_0^2}, \quad (25)$$

$$\frac{\tilde{d\epsilon}}{dt} = \frac{q}{m}\vec{v}_{g0} \cdot \vec{E} + \mu\frac{\partial \tilde{B}}{\partial t}, \quad (26)$$

$$\tilde{B} = \vec{e}_{\parallel 0} \cdot \vec{\tilde{B}} \quad \text{and} \quad \vec{\tilde{e}}_\parallel = \frac{\vec{\tilde{B}}_\perp}{B_0}. \quad (27)$$

Equation(24) is again solved with a perturbation method. The operator

$$\vec{v}_d \cdot \nabla = \mathcal{O}(\varepsilon) \quad (28)$$

on the left hand side of Eq.(24) and the parameter

$$b_p = \mathcal{O}(\varepsilon), \tag{29}$$

are considered as perturbative, ε being defined as the small parameter of the perturbation. The spirit of our calculation is to retain only the most dominant element among comparable terms. In this way, and for realistic values, we can actually neglect the contribution (22) of order $\mathcal{O}(\lambda)$ to f_0 on the right hand side of Eq.(24). As the unperturbed system is homogeneous in time and the toroidal direction, one can consider perturbations whose dependence in time and the angle φ is of the form

$$\exp i\left[n\varphi - (\omega + i\eta)t\right], \tag{30}$$

where n is the toroidal mode number and $i\eta, \eta > 0$, a small imaginary part that is added to the frequency so as to define causality. By writting $\tilde{f}$ in a series expansion with respect to ε

$$\tilde{f} = \tilde{f}^{(0)} + \tilde{f}^{(1)} + \mathcal{O}(\varepsilon^2) \tag{31}$$

and retaining only the terms of order $\mathcal{O}(\varepsilon)$ in Eq.(24) leads to

$$i\Omega_0 \tilde{f}^{(0)} = \left(\tilde{\vec{v}}_g \cdot \frac{\partial}{\partial \vec{X}} + \frac{\tilde{d\epsilon}}{dt} \frac{\partial}{\partial \epsilon} \right) F, \tag{32}$$

where $\Omega_0 = \omega + i\eta - k_\varphi v_{||}$ and $k_\varphi = n/r$. Let us define the coefficients

$$\beta_{\mu,\nu} = \vec{e}_\nu \cdot \nabla \times \vec{e}_\mu \qquad\qquad \mu, \nu \in \{n, b, ||0\}, \tag{33}$$

characterizing the geometry. On using Faraday's law to express $\tilde{\vec{B}}$ in terms of $\vec{E}$, the relations

$$\vec{e}_{||0} \cdot (\nabla \vec{e}_{||0}) \cdot \vec{e}_n = \beta_{||b}, \qquad\qquad \vec{e}_{||0} \cdot (\nabla \vec{e}_{||0}) \cdot \vec{e}_b = -\beta_{||n}, \tag{34}$$

as well as the following relations valid to order $\mathcal{O}(\lambda)^1$

$$\nabla_n \ln \Omega = \beta_{||b} \quad \text{and} \quad \nabla_p \ln \Omega = -\beta_{||n}, \tag{35}$$

the solution of Eq.(32) is found to be

$$\begin{aligned}\tilde{f}^{(0)} &= \frac{-iq}{m\Omega\Omega_0}\left\{\frac{1}{\omega}\left(\Omega_0 E_b - iv_{\|}\nabla_p E_{\|}\right)\nabla_n F\right. \\ &\left.+\left[\Omega v_{\|}E_{\|} + \left(\frac{v_\perp^2}{2}+v_{\|}^2\right)\vec{\beta_\perp}\cdot\vec{E} - i\omega\frac{v_\perp^2}{2}\tilde{B}\right]\frac{\partial F}{\partial\epsilon}\right\},\end{aligned} \tag{36}$$

where we have used the notations

$$\tilde{B} = \frac{1}{i\omega}\left(\nabla_n E_b - \nabla_p E_n + \beta_{n\|}E_n + \beta_{b\|}E_b\right), \tag{37}$$

$$\vec{\beta_\perp} = \left(\nabla\times\vec{e}_{\|0}\right)_\perp = \beta_{\|n}\,\vec{e_n} + \beta_{\|b}\,\vec{e_b}, \tag{38}$$

$$\nabla_\nu = \vec{e_\nu}\cdot\nabla. \tag{39}$$

Notice that the first term in Eq.(36) actually does not have any resonant denominator Ω_0 and therefore does not contribute to the absorbed power. In our approach we write Eq.(24) to order $\mathcal{O}(\varepsilon^2)$ retaining only this term in $\tilde{f}^{(0)}$ so as to obtain an associated non-vanishing contribution to the power. Writting Eq.(24) to second order in ε thus reduces to

$$i\Omega_0\,\tilde{f}^{(1)} + \left(v_{\|}b_p\nabla_p + \vec{v_d}\cdot\nabla\right)\left(\frac{iq}{m\omega\Omega}E_b\nabla_n F\right) = ib_p\frac{v_{\|}}{\omega B_0}\nabla_p E_b\nabla_n F \tag{40}$$

and the contribution to $\tilde{f}$ in first order becomes

$$\tilde{f}^{(1)} = -\frac{q}{m\omega\Omega^2\Omega_0}\left(\frac{v_\perp^2}{2}+v_{\|}^2\right)(\vec{\beta_\perp}\cdot\nabla)E_b\nabla_n F. \tag{41}$$

Let us remark that this method of calculation was partly inspired by a similar method used by Rosenbluth and Rutherford in the special case of Alfvén waves [4].

III. Derivation of the power absorption

A local energy conservation law can be derived from Eq.(5)

$$\frac{\partial}{\partial t}\int\frac{1}{2}m(v_{\|}^2+v_\perp^2)fd^3v + \nabla\cdot\int\vec{v_g}\,\frac{1}{2}m(v_{\|}^2+v_\perp^2)fd^3v = \int m\frac{d\epsilon}{dt}fd^3v, \tag{42}$$

using the incompressibility of the "phase space" flow

$$\nabla\cdot\vec{v_g} + \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left(v_\perp\frac{dv_\perp}{dt}\right) + \frac{\partial}{\partial v_{\|}}\frac{dv_{\|}}{dt} = \mathcal{O}(\lambda^2), \tag{43}$$

and the notation $d^3v = 2\pi v_\perp dv_\perp dv_\parallel$. The corresponding global conservation law is given by

$$\int \frac{1}{2} m(v_\parallel^2 + v_\perp^2)\frac{\partial f}{\partial t} d\Gamma = \int m \frac{d\epsilon}{dt} f d\Gamma, \tag{44}$$

where $d\Gamma = d^3X d^3v$. The left hand side of Eq.(44) stands for the variation with respect to time of the total kinetic energy of the particles. Using the complex notation (30), the total power exchanged between the particles and the perturbing EM fields averaged over time reads

$$\bar{P} = \frac{1}{2}\Re e \int d\Gamma \, \frac{\widetilde{d\epsilon}}{dt} \tilde{f}^* \, . \tag{45}$$

Inserting relations (26), (36) and (41) in (45) allows us to express the power in terms of the electric field. One can decompose the power as follows

$$\bar{P} = \bar{P}_{homo} + \bar{P}_{inhomo}, \tag{46}$$

$$\bar{P}_{homo} = \frac{1}{2}\Im m \int d\Gamma \frac{q^2}{m\Omega_0}\left| v_\parallel E_\parallel + \frac{1}{\Omega}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)(\vec{\beta}_\perp \cdot \vec{E}) - i\frac{\omega}{\Omega}\frac{v_\perp^2}{2}\tilde{B}\right|^2 \frac{\partial F}{\partial \epsilon}, \tag{47}$$

$$\begin{aligned}\bar{P}_{inhomo} &= -\frac{1}{2}\Re e \int d\Gamma \frac{q^2}{m\omega\Omega\Omega_0}\left[v_\parallel E_\parallel + \frac{1}{\Omega}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\vec{\beta}_\perp \cdot \vec{E} - i\frac{\omega}{\Omega}\frac{v_\perp^2}{2}\tilde{B}\right] \times \\ &\left[\frac{1}{\Omega}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)(\vec{\beta}_\perp \cdot \nabla)E_b^* + v_\parallel \nabla_p E_\parallel^*\right] \nabla_n F.\end{aligned} \tag{48}$$

Eq.(47) shows that $\bar{P}_{homo}$ is positive definite with respect to $\vec{E}$. For this reason, the integrand of this relation can actually be considered as a local power absorption density in "phase space". This is not the case for $\bar{P}_{inhomo}$, which contains the terms relative to the gradients of the distribution function. For this reason, instabilities can only arise due to gradients in density or in temperature of the plasma. The resonant denominator can be written as

$$\frac{1}{\Omega_0} = \frac{1}{\omega - k_\varphi v_\parallel + i\eta} = \mathcal{P}\frac{1}{\omega - k_\varphi v_\parallel} - i\pi\frac{1}{|k_\varphi|}\delta\left(v_\parallel - \frac{\omega}{k_\varphi}\right), \tag{49}$$

where $\mathcal{P}$ stands for the principle value and δ for the Dirac function. Globally, only the resonant particles, that is, whose velocity is such that $v_\parallel$ is equal to the phase velocity $v_p = \omega/k_\varphi$, can exchange energy with the EM fields and contribute to the power. The principle value in relation (49) can therefore be discarded when evaluating (48). For a Maxwellian distribution of the form (17), we get

$$\begin{aligned}\bar{P}_{homo} &= \sqrt{\pi}\epsilon_0 \int d^3x \frac{\omega_p^2 v_{th}}{4\Omega^2 |k_\varphi|} \exp(-z_0^2) \times \\ & \left\{ \left| 2z_0 \frac{\Omega}{v_{th}} E_\| + \left(1+2z_0^2\right) \vec{\beta}_\perp \cdot \vec{E} - i\omega \tilde{B} \right|^2 + \left| \vec{\beta}_\perp \cdot \vec{E} - i\omega \tilde{B} \right|^2 \right\},\end{aligned} \tag{50}$$

$$\begin{aligned}\bar{P}_{inhomo} &= \sqrt{\pi}\epsilon_0 \Im m \int d^3x \nabla'_n \frac{\omega_p^2 v_{th}^3}{8\omega\Omega^3 |k_\varphi|} \exp(-z_0^2) \left\{ \left[2z_0 \frac{\Omega}{v_{th}} E_\| + \left(1+2z_0^2\right) \vec{\beta}_\perp \cdot \vec{E} - i\omega \tilde{B} \right] \times \right. \\ & \left. \left[\left(1+2z_0^2\right) (\vec{\beta}_\perp \cdot \nabla E_b^*) + 2z_0 \frac{\Omega}{v_{th}} \nabla_p E_\|^* \right] + \left[\vec{\beta}_\perp \cdot \vec{E} - i\omega \tilde{B} \right] (\vec{\beta}_\perp \cdot \nabla) E_b^* \right\},\end{aligned} \tag{51}$$

where $z_0 = \omega / k_\varphi v_{th}$, ϵ_0 is the permittivity of free space , $\omega_p^2 = Nq^2/m\epsilon_0$ the plasma frequency squared and ∇'_n is equivalent to ∇_n except that it operates only on density and temperature. For the slowing-down distribution (18) resonant particles can exist only if $|v_p| < v_0$, in that case the power reads

$$\begin{aligned}\bar{P}_{homo} &= \pi^2 \epsilon_0 \int d^3x \frac{\omega_p^2 C}{|k_\varphi|} \left\{ \frac{1}{|v_p|^3 + v_c^3} \left| v_p E_\| + \frac{1}{\Omega} v_p^2 \vec{\beta}_\perp \cdot \vec{E} \right|^2 \right. \\ & \left. + \frac{2I_0}{\Omega} \Re e \left(\vec{\beta}_\perp \cdot \vec{E}^* + i\omega \tilde{B}^* \right) \left(v_p E_\| + \frac{v_p^2}{\Omega} \vec{\beta}_\perp \cdot \vec{E} \right) + \frac{I_1}{\Omega^2} \left| \vec{\beta}_\perp \cdot \vec{E} - i\omega \tilde{B} \right|^2 \right\},\end{aligned} \tag{52}$$

$$\begin{aligned}\bar{P}_{inhomo} &= -\pi^2 \epsilon_0 \Im m \int d^3x \nabla'_n \frac{\omega_p^2 C}{|k_\varphi| \Omega} \left\{ I_0 \left(v_p E_\|^* + \frac{v_p^2}{\Omega} \vec{\beta}_\perp \cdot \vec{E}^* \right) \left(\frac{v_p^2}{\omega\Omega} (\vec{\beta}_\perp \cdot \nabla) E_b + \frac{1}{k_\varphi} \nabla_p E_\| \right) \right. \\ & + \frac{I_1}{2\Omega} \left[\left(\vec{\beta}_\perp \cdot \vec{E}^* + i\omega \tilde{B}^* \right) \left(\frac{v_p^2}{\omega\Omega} (\vec{\beta}_\perp \cdot \nabla) E_b + \frac{1}{k_\varphi} \nabla_p E_\| \right) \right. \\ & \left. \left. + \frac{1}{\omega} (\vec{\beta}_\perp \cdot \nabla) E_b \left(v_p E_\|^* + \frac{v_p^2}{\Omega} \vec{\beta}_\perp \cdot \vec{E}^* \right) \right] + \frac{I_2}{4\omega\Omega^2} (\vec{\beta}_\perp \cdot \nabla) E_b \left(\vec{\beta}_\perp \cdot \vec{E}^* + i\omega \tilde{B}^* \right) \right\},\end{aligned} \tag{53}$$

where

$$I_n = \int_{|v_p|}^{v_0} dv \frac{v \left(v^2 - v_p^2\right)^n}{v^3 + v_c^3}. \tag{54}$$

These integrals, although somewhat lengthy, can easily be evaluated using the notations

$$z_3 = a + x^3, \qquad \alpha = a^{1/3}, \tag{55}$$

and the following relations

$$\begin{aligned}
\int \frac{x^n dx}{z_3} &= \frac{x^{n-2}}{(n-2)} - a\int \frac{x^{n-3}dx}{z_3}, \\
\int \frac{dx}{z_3} &= \frac{\alpha}{3a}\left\{\frac{1}{2}\ln\frac{(x+a)^2}{x^2-\alpha x+\alpha^2} + \sqrt{3}\arctan\frac{2x-\alpha}{\alpha\sqrt{3}}\right\}, \\
\int \frac{xdx}{z_3} &= -\frac{1}{3\alpha}\left\{\frac{1}{2}\ln\frac{(x+a)^2}{x^2-\alpha x+\alpha^2} - \sqrt{3}\arctan\frac{2x-\alpha}{\alpha\sqrt{3}}\right\}, \\
\int \frac{x^2dx}{z_3} &= \frac{1}{3}\ln z_3.
\end{aligned} \tag{56}$$

The relations (50)-(53) can be implemented in a numerical code using a given perturbing electrical field $\vec{E}$. This approach is valid as long as the corresponding damping rate remains small compared to the frequency ω, otherwise the EM fields would have to be evaluated self-consistently.

IV. Elimination of $E_{\parallel}$

If the EM fields are provided by an ideal-MHD calculation, the component of $\vec{E}$ parallel to the magnetostatic field is zero and must therefore be obtained from a more general model. To do this, we restrict ourselves to the Alvén wave range of frequencies, so that

$$v_p \sim c_A, \tag{57}$$

where c_A is the Alfvén velocity, and $c_A \ll c$ one of the validity conditions of the MHD model. Using the quasi-neutrality condition valid in the frame of the MHD model

$$\widetilde{N}_e = \widetilde{N}_i, \tag{58}$$

$\widetilde{N}_e$ and $\widetilde{N}_i$ being the perturbed densities of the electrons and the ions, and expressing these densities in terms of the electrical field allows one to write $E_{\parallel}$ in terms of $\vec{E}_{\perp}$. Due to the fact that the thermal velocity v_{the} of the electrons can be comparable to the phase velocity v_p

$$v_{the} \sim v_p, \tag{59}$$

it is reasonable to use the solution (36) of the DKE to evaluate the dominant term of $\tilde{N}_e$

$$\begin{aligned}\tilde{N}_e &= \int d^3v\, \tilde{f}_{electron} \cong \frac{ie}{m_e\Omega_e}\int \frac{d^3v}{\Omega_0}\left[\Omega_e v_\| E_\| + \left(\frac{v_\perp^2}{2}+v_\|^2\right)\vec{\beta_\perp}\cdot\vec{E}_\perp - i\omega\frac{v_\perp^2}{2}\tilde{B}\right]\frac{\partial F}{\partial\epsilon} \qquad (60)\\ &= 2i\frac{eN_e}{m_e\omega\Omega_e}\left[\frac{\omega\Omega_e}{k_\varphi v_{the}^2}(1-Z)E_\| - \frac{1}{2}Z\left(\vec{\beta_\perp}\cdot\vec{E}_\perp - i\omega\tilde{B}\right) + \left(\frac{\omega}{k_\varphi v_{the}}\right)^2(1-Z)\vec{\beta_\perp}\cdot\vec{E}\right],\end{aligned}$$

where

$$Z \equiv Z\left(\frac{\omega}{k_\varphi v_{the}}\right) \quad \text{and} \quad Z(z) = \frac{z}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{1}{z-x}\exp(-x^2)dx, \Im m z > 0. \qquad (61)$$

To evaluate $\tilde{N}_i$ one can use a cold fluid model for the ions as the phase velocity v_p is much higher than the thermal velocity v_{thi} of the ions. Due to their relatively large mass, the ions dominate the motion perpendicular to the magnetostatic field. Solving the equation of motion in the perpendicular plane

$$-i\omega m_i\vec{v} = e\left(\vec{E}_\perp + \vec{v}\times\vec{B_0}\right), \qquad (62)$$

one obtains for low frequencies $\omega \ll \Omega_i$

$$\vec{v} = \frac{e}{m_i\Omega_i}\left[\vec{E}_\perp\times\vec{e_{\|0}} - i\frac{\omega}{\Omega_i}\vec{E}_\perp\right]. \qquad (63)$$

Using the equation of continuity for the ion density leads to

$$\tilde{N}_i = -\frac{i}{\omega}\nabla\cdot\left(N_i\vec{v}\right) \cong -\frac{eN_i}{m_i\Omega_i^2}\left[\nabla\cdot\vec{E}_\perp - 2i\frac{\Omega_i}{\omega}\vec{\beta_\perp}\cdot\vec{E}_\perp - \Omega_i\tilde{B}\right] \qquad (64)$$

after having kept the most dominant terms and used Eq.(35). Inserting Eqs.(60) and (64) in Eq.(58) finally leads to

$$E_\| = -\frac{k_\varphi v_{the}^2}{2\omega\Omega_e}\left\{\frac{1}{1-Z}\left(i\frac{\omega}{\Omega_i}\nabla\cdot\vec{E}_\perp + \vec{\beta_\perp}\cdot\vec{E}_\perp\right) + \left[1+2\left(\frac{\omega}{k_\varphi v_{the}}\right)^2\right]\vec{\beta_\perp}\cdot\vec{E}_\perp - i\omega\tilde{B}\right\}, \qquad (65)$$

where we have used the neutrality of the stationary state. This relation can now be inserted in Eq (50). For electrons and ions one obtains respectively

$$\bar{P}_{homo} = \sqrt{\pi}\epsilon_0\int d^3x\frac{\omega_p^2 v_{th}}{4\Omega^2|k_\varphi|}\exp -z_0^2\times\left(|a_{species}|^2 + \left|\vec{\beta_\perp}\cdot\vec{E} - i\omega\tilde{B}\right|^2\right), \qquad (66)$$

$$a_{electron} = \frac{1}{1-Z}\left(i\frac{\omega}{\Omega_i}\nabla\cdot \vec{E}_\perp + \vec{\beta}_\perp \cdot \vec{E}_\perp\right), \tag{67}$$

$$\begin{aligned} a_{ion} &= \frac{T_e}{T_i}\frac{1}{1-Z}\left(i\frac{\omega}{\Omega_i}\nabla\cdot \vec{E}_\perp + \vec{\beta}_\perp \cdot \vec{E}_\perp\right) - i\omega\left(1+\frac{T_e}{T_i}\right)\tilde{B} \\ &+ \left[1+\frac{T_e}{T_i}+2\left(\frac{\omega}{k_\varphi v_{thi}}\right)^2\right]\vec{\beta}_\perp \cdot \vec{E}_\perp . \end{aligned} \tag{68}$$

$\bar{P}_{inhomo}$ can be neglected for these species. This is related to the fact that in the frame of the present work $\omega \gg \omega^\star$ for electrons and ions, where

$$\omega^\star = \frac{|\nabla\psi|}{N}\frac{dN}{d\psi}\frac{T}{m\Omega}k_p \tag{69}$$

is the characteristic frequency of the drift mode, k_p being the poloidal wave number. Let us note that in Eq.(68) the contributions of $E_\parallel$ to the power absorbed by the ions are proportional to T_e/T_i. As the average kinetic energy of the alpha particles is very high as compared with the energy of the electrons, the contributions of $E_\parallel$ to the power absorbed by the alphas can therefore be neglected. In this way, one can write for the alpha particles

$$\begin{aligned} \bar{P}^{\alpha}_{homo} &= \pi^2\epsilon_0 \int d^3x \frac{\omega^2_{p\alpha}C}{|k_\varphi|\Omega^2_\alpha}\left\{\left[\frac{v_p^4}{|v_p|^3+v_c^3}+2v_p^2 I_0\right]\left|\vec{\beta}_\perp\cdot\vec{E}\right|^2\right. \\ & \left. 2\omega v_p^2 I_0 \Im m\left(\tilde{B}\vec{\beta}_\perp\cdot\vec{E}^\star\right) + I_1\left|\vec{\beta}_\perp\cdot\vec{E} - i\omega\,\tilde{B}\right|^2\right\}, \end{aligned} \tag{70}$$

$$\begin{aligned} \bar{P}^{\alpha}_{inhomo} &= \frac{\pi^2\epsilon_0}{\omega}\Im m \int d^3x \nabla'_n \frac{\omega^2_{p\alpha}C}{|k_\varphi|\Omega^3_\alpha}\left\{\left(v_p^4 I_0 + v_p^2\frac{I_1}{2}\right)\vec{\beta}_\perp\cdot\vec{E}\right. \\ & \left.\left(v_p^2\frac{I_1}{2}+\frac{I_2}{4}\right)\left(\vec{\beta}_\perp\cdot\vec{E} - i\omega\,\tilde{B}\right)\right\}(\vec{\beta}_\perp\cdot\nabla)E_b^\star. \end{aligned} \tag{71}$$

Relations (67)-(71) are now ready for implementation in an ideal-MHD code.

V. Conclusions

Explicit relations for the power absorbed by different species have been obtained in terms of given small amplitude, low frequency EM fields in a hot, toroidal, axisymmetric plasma using the DKE theory. These results are ready for implementation in a numerical code and in particular for the study of Alfvén waves using EM fields obtained from ideal-MHD computations.

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